Explicit bounds for the height of the modular polynomials Φ_N

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joint with Florian Breuer and Fabien Pazuki

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Introduction and previous results

- **j-invariant**
  - Classifying invariant for elliptic curves over $\overline{\mathbb{Q}}$.
  - Modular function $j : \mathbb{H} \to \mathbb{C}$ for $\mathrm{SL}_2(\mathbb{Z})$, with q-expansion

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Properties

- Coefficients in $\mathbb{Z}$, $\Phi_N(X, Y) = \Phi_N(Y, X)$, irreducible over $\mathbb{Q}$
- (Dedekind psi function) $\deg_X \Phi_N = \psi(N) := N \prod_{p|N} \left(1 + \frac{1}{p}\right) = O(N \log\log N)$.
- $\Phi_N(X, Y) = 0$ is a plane affine integral model of the modular curve $X_0(N)$.

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- **Mahler [Mah74]**
  - $h(\Phi_{2^n}) = \mathrm{O}(2^n n)$
  - $h(\Phi_N) = \mathrm{O}(N^{3/2})$
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  - **CONJECTURE:** $h(\Phi_N) = \mathrm{O}(N \log N \log\log N)$.
- **Tretkoff [Coh84]** for *any* $N \in \mathbb{Z}_{\geq 1}$, where the $\mathrm{O}(1)$ term was not made *explicit*:

$$h(\Phi_N) = 6\psi(N)\left(\log N - 2\kappa(N) + \mathrm{O}(1)\right),$$

$$\text{for } \kappa(N) = \sum_{p|N} \frac{\log p}{p} = \mathrm{O}(\log\log N).$$

Explicit results

- (**Bröker-Sutherland ([BrSu10])**) Explicit for ℓ prime:

$$h(\Phi_\ell) \leq 6\ell \log \ell + 16\ell + 14\sqrt{\ell} \log \ell.$$

- (**Breuer-Pazuki [BP24]**) Explicit in general,

$$h(\Phi_N) \leq 6\psi(N) \left(\log N - 2\lambda(N) + \log\log N + 4.436\right),$$

with $\kappa(N) = \sum_{p|N} \frac{1}{p} \log p$
changed to $\lambda(N) = \sum_{p^n \| N} \frac{p^n - 1}{p^{n-1}(p^2-1)} \log p$.

Our result

Theorem(Breuer-Pazuki-G. [BGP25])

For any $N \in \mathbb{Z}_{\geq 1}$,

$$h(\Phi_N) \leq 6\psi(N)\big(\log N - 2\lambda(N) + 9.5387\big),$$
$$6\psi(N)\big(\log N - 2\lambda(N) - 0.0351\big) \leq h(\Phi_N)$$

Comments

- Compare with [Paz19], which admits a generalization to higher dimensions in [Kief22]: It gets $\mathrm{O}(\psi(N) \log N)$, but not the right constant 6.

$$h(\Phi_N) \leq \psi(N)[6 \log N + \log \psi(N) + 6 \log(12 \log N + 2 \log \psi(N) + 25.2) + 15.7]$$

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- This approach is specific to the properties of the j-invariant and $\mathrm{SL}_2(\mathbb{Z})$.
- It has been extended to Drinfeld modular polynomial in [BPR24].

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- Key formula
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We set $\tau = iy$ such that $1728 \leq j(\tau) \leq 2 \cdot 1728$ (so $y = \mathrm{Im}(\tau) \in [1, 1.2536]$).

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$$C_N := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \ : \ ad = N, \ 1 \leq a, \ 0 \leq b \leq d-1, \ \gcd(a,b,d) = 1 \right\};$$

$$S_N(\tau) := m(\Phi_N(X, j(\tau))) = \sum_{\gamma \in C_N} \log\max\{1, |j(\tau_\gamma)|\}, \ \tau_\gamma := \gamma\tau = \frac{a_\gamma \tau + b_\gamma}{d_\gamma}.$$

Lemma ([BrSu10])

$$h(\Phi_N) \le \max_{L \le j(\tau) \le 2L} S_N(\tau) + \psi(N)\left(\frac{1+\log L}{L} + 4\log 2\right).$$

Lemma ([BrSu10])

$$h(\Phi_N) \leq \max_{L \leq j(\tau) \leq 2L} S_N(\tau) + \psi(N) \left(\frac{1 + \log L}{L} + 4 \log 2 \right).$$

Definitions

- $\mathcal{F}$: standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$.
- (reduced) $\widetilde{\omega} \in \mathrm{SL}_2(\mathbb{Z}) \cdot \omega \cap \mathcal{F}$.

Key formula (for any τ) [Aut03], [BP24]:

- $\log\max\{1, |j(\tau)|\} \rightsquigarrow \log\max\{|\Delta(\tau)|, |j(\tau)\Delta(\tau)|$ (bounded in the fundamental domain $\mathcal{F}$), and reduce τ_γ to $\widetilde{\tau}_\gamma \in (\mathrm{SL}_2(\mathbb{Z}) \cdot \tau) \cap \mathcal{F}$.

[Aut03]

$$\prod_{\gamma \in C_N} \Delta(\tau_\gamma) = (-\Delta(\tau))^{\psi(N)};$$

$$\sum_{\gamma \in C_N} \log \frac{d_\gamma}{a_\gamma} = \psi(N)\left(\log(N) - 2\lambda(N)\right);$$

$$-\sum_{\gamma \in C_N} \log \mathrm{Im}(\tau_\gamma) = \psi(N)\left(\log N - 2\lambda(N) - \log \mathrm{Im}(\tau)\right).$$

$$S_N(\tau) = \sum_{\gamma \in C_N} \log \max\{1, |j(\tau_\gamma)|\}$$

$$= \sum_{\gamma \in C_N} \log \max\{|\Delta(\tau_\gamma)|, |\Delta(\tau_\gamma)| |j(\tau_\gamma)|\} - \sum_{\gamma \in C_N} \log |\Delta(\tau_\gamma)|$$

$$= \sum_{\gamma \in C_N} \log \max\{|\Delta(\widetilde{\tau_\gamma})|, |\Delta(\widetilde{\tau_\gamma})| |j(\tau_\gamma)|\} + 6 \sum_{\gamma \in C_N} (\log \operatorname{Im}(\widetilde{\tau_\gamma}) - \log \operatorname{Im}(\tau_\gamma))$$

$$- \sum_{\gamma \in C_N} \log |\Delta(\tau_\gamma)|.$$

$$S_N(\tau) = \sum_{\gamma \in C_N} \log \max\{|\Delta(\widetilde{\tau_\gamma})|, |j(\widetilde{\tau_\gamma})\Delta(\widetilde{\tau_\gamma})|\} + 6\psi(N)\big(\log N - 2\lambda_N\big)$$
$$+ 6 \sum_{\gamma \in C_N} \log \operatorname{Im} \widetilde{\tau_\gamma} - \psi(N) \log\left[|\Delta(\tau)|(\operatorname{Im}\tau)^6\right].$$

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Lemma

Have the following bounds on $\mathcal{F}$:

$$-5.5335 < \log \max\{|\Delta(\widetilde{\tau_\gamma})|, |j(\widetilde{\tau_\gamma})\Delta(\widetilde{\tau_\gamma})|\} < 1.1266$$

"Small divisors"

Problem: Understand the reduction of the isogenies modulo $\mathrm{SL}_2(\mathbb{Z}) \rightsquigarrow$ Understand the distribution of the isogenies on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

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Direct case If $\mathrm{Im}(\tau_\gamma) > 1$ then $\mathrm{Im}(\widetilde{\tau_\gamma}) = \mathrm{Im}(\tau_\gamma)$.

$$1 < \mathrm{Im}(\tau_\gamma) = \frac{a}{d}y = \frac{N}{d^2}y \iff d < \sqrt{Ny}$$

"Large divisors"

Example For $N = p$ prime, then $d \in \{1, p\}$, and the isogenies

- pyi,
- $\frac{b}{p} + \frac{y}{p} i$, for $0 \leq b \leq p - 1$.

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Need to reduce the second ones modulo $\mathrm{SL}_2(\mathbb{Z})$.
First intuition (wrong). For a given rational numbers h/k, $h/k \in \mathrm{SL}_2(\mathbb{Z})\infty$ with matrix

$$\delta = \begin{pmatrix} * & * \\ k & -h \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \; \tau \mapsto \frac{*}{k\tau - h}$$

So if ω is "close" to a rational number, via this fractional transformation above, $\delta(\omega)$ should be send "close" to ∞.

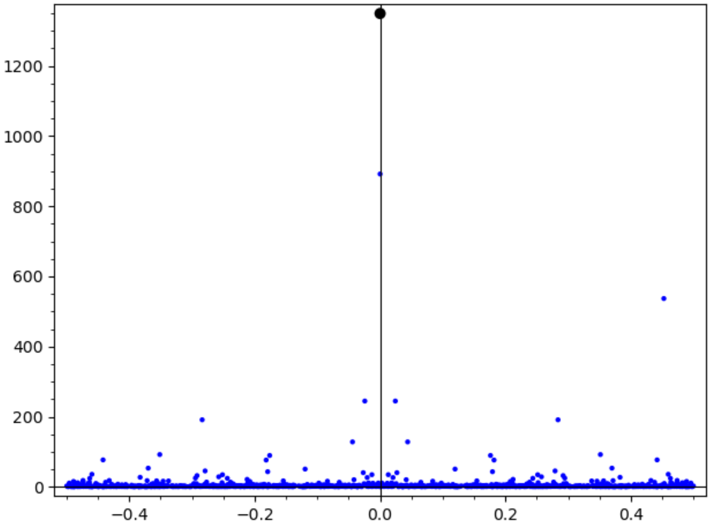

Figure: Reduced isogenies for $\tau = 1.23i$, $p = 1097$

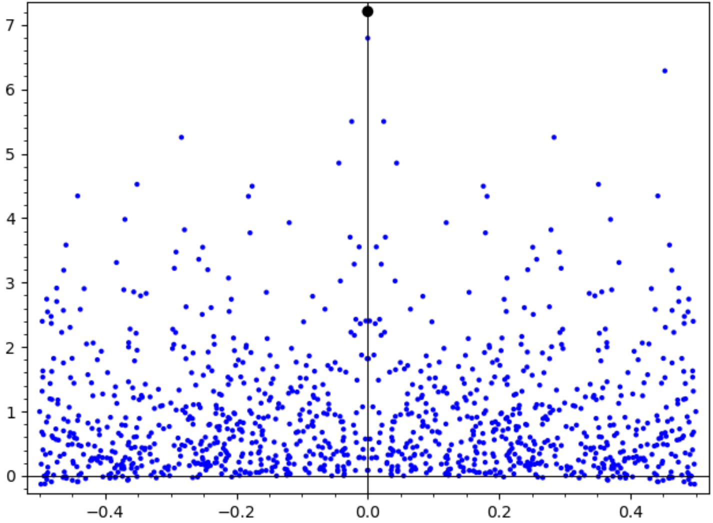

Figure: Reduced isogenies for $\tau = 1.23i$, $p = 1097$, plotting $\log \operatorname{Im}$

The isogenies actually *equidistribute* (with respect to the hyperbolic metric $\frac{ds^2}{(\operatorname{Im}\tau)^2}$). Easier to visualize with another fundamental domain

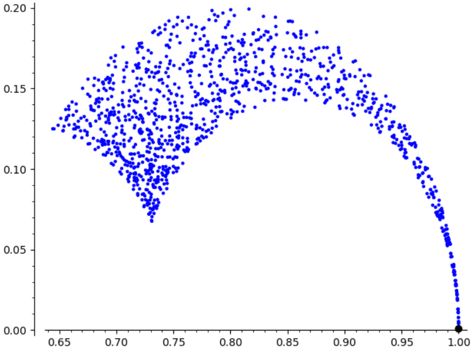

Figure: Reduced isogenies for $\tau = 1.23i$, $p = 1097$, plotting $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \mathcal{F}$

Upper bound

Second intuition (less wrong) For $r = h/k$ and the matrix $\delta = \begin{pmatrix} * & * \\ k & -h \end{pmatrix}$, then

$$\mathrm{Im}(\delta\omega) = \frac{1}{|k\omega - h|^2}\,\mathrm{Im}(\tau) = \frac{1}{k^2|\omega - \frac{h}{k}|^2}\,\mathrm{Im}(\omega) = \frac{1}{den(r)^2}\frac{1}{|\omega - r|^2}\,\mathrm{Im}(\omega)$$

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In our example, for $r = \frac{b}{p}$, $\tau_\gamma = \frac{b}{p} + \frac{y}{p}i$, actually

$$\frac{1}{p^2}\frac{1}{|\frac{y}{p}|^2} = \frac{1}{y^2} \approx 1.$$

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Need to compare τ_γ with a rational number of **small denominator**.

Assuming $d \geq \sqrt{Ny}$, set $M := \left\lfloor \frac{d}{\sqrt{Ny}} \right\rfloor \in \mathbb{Z}_{\geq 1}$. Consider the interval of length 1

$$I_M = \left[\frac{1}{M+1}, 1 + \frac{1}{M+1}\right)$$

Farey's decomposition of the unit interval

Have a partition of I_M in subintervals

$$I_M = \bigcup_{k=1}^{M} \bigcup_{\substack{h=1 \\ (h,k)=1}}^{k} I_M\left(\frac{h}{k}\right),$$

and for each $I_M\left(\frac{h}{k}\right) = [\rho_1, \rho_2]$ have

$$\frac{1}{2Mk} \leq \frac{h}{k} - \rho_1 \leq \frac{1}{(M+1)k}$$
$$\frac{1}{2Mk} \leq \rho_2 - \frac{h}{k} \leq \frac{1}{(M+1)k}$$

Algorithm: For every d "large divisor" of N, set $M = M(d)$ and I_M as above. Consider $\tau_\gamma = \frac{b}{d} + \frac{a}{d}yi$, then $b/d \in [0, 1)$.

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- Consider the unique h'/k' such that $\frac{b}{d} \in I_M\left(\frac{h'}{k'}\right)$.
- By Bezout's lemma, consider (r,s) with $r(-h) + sk = 1$.
- Set $\delta = \begin{pmatrix} r & s \\ k & -h \end{pmatrix}$.
- If $|Re(\delta\tau_\gamma)| > 1/2$, consider suitable integer $n \in \mathbb{Z}$ so that $|Re(\delta\tau_\gamma - n)| \le 1/2$ and set

$$\widehat{\tau_\gamma} = \delta\tau_\gamma - n.$$

Proposition

Have the following properties for $\delta\tau_\gamma$.

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- $$\log \widehat{\tau_\gamma} \leq \log\left(\frac{d^2}{Nyk^2}\right).$$

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From the bound $\log \widehat{\tau_\gamma} \leq \log\left(\frac{d^2}{Nyk^2}\right) \approx \log\left(\frac{M^2}{k}\right)$, (recall $M \approx d/\sqrt{Ny}$).

- if $k = 1$ then the upper bound is $\log M^2$, and for $d = N$ it becomes $\log M^2 \approx \log(N/y)^2$, which is large.
- if $k = M$, then $\log(1 + o(1))^2$, so it is bounded.

Example For $p = 31$ and $\tau = 1.23i$, then $M(p) = 5$.

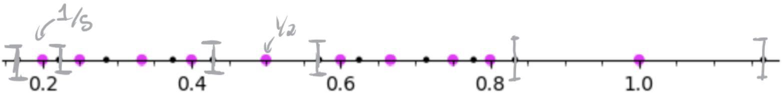

Figure: Farey decomposition of level 5. The pink points are h/k with $1 \leq k \leq 5$ and $1 \leq h \leq k$ with $(h,k) = 1$. The black points are the extremes of $I_M(h/k)$. Recall $\frac{2}{2Mk} \leq length(I_M(h/k)) \leq \frac{2}{(M+1)k}$.

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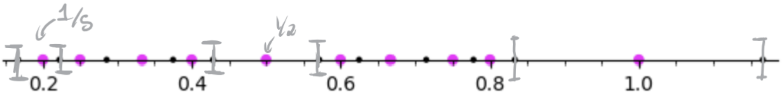

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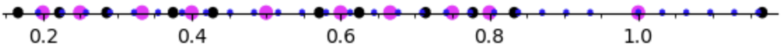

Figure: The blue points are $\frac{b}{p}$ or $\frac{b+p}{p}$ with $0 \leq b \leq p-1$.

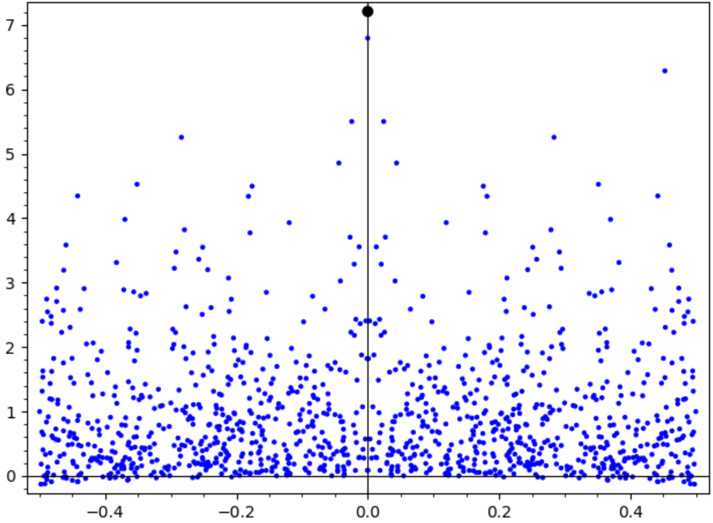

Figure: Reduced isogenies for $\tau = 1.23i$, $p = 1097$, plotting $\log \operatorname{Im}$

Conclusion: Can bound the sums

- Small divisors $d < \sqrt{Ny}$:

$$\sum_{\substack{\gamma \in C_N \\ d < \sqrt{Ny}}} \log \operatorname{Im} \widetilde{\tau_\gamma} \leq \psi(N) \left(\frac{1}{e} + \log y \right) \leq 0.575 \psi(N)$$

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- Large divisors $d \geq \sqrt{Ny}$

$$\sum_{\substack{\gamma \in C_N \\ d \geq \sqrt{Ny}}} \log \operatorname{Im} \widetilde{\tau_\gamma} \leq \left(6.83 + \frac{0.5 + \log 2}{2\sqrt{N}} \right) \psi(N) \leq_{(N > 400)} 7.206 \psi(N).$$

Lower bound

- $h(\Phi_N) \geq h(\Phi_N(X,0))$ trivially; and $j(\rho) = 0$ with $\rho = e^{i\frac{\pi}{3}}$
- Can compare $h(\Phi_N(X,0))$ with $m(\Phi_N(X,0))) = S_N(\rho)$:

$$S_N(\rho) \leq \log(\Phi(N) + 1) + h(\Phi_N).$$

- Can get lower bounds in the key formula, and for the sum $\sum_{C_N} \log \operatorname{Im} \widetilde{\tau_\gamma}$ use $\operatorname{Im} \widetilde{\tau_\gamma} \geq \frac{\sqrt{3}}{2}$.

Thank you!
Merci !
¡Gracias!

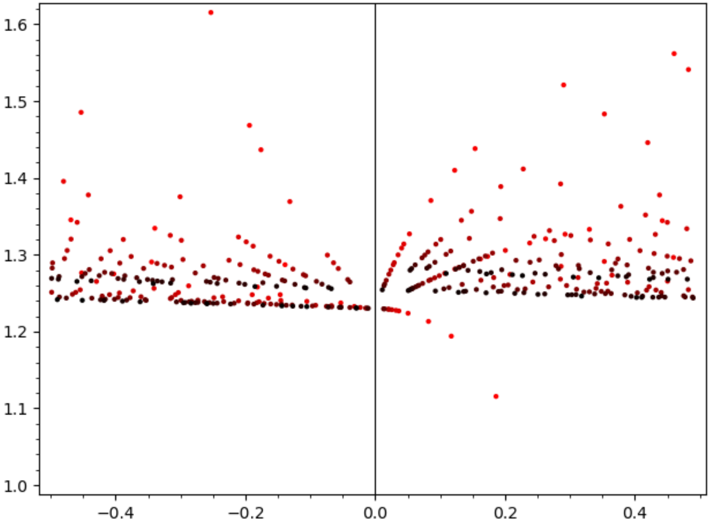

Figure: For p varying over the first 500 primes and $\tau = 1.23i$, it plots the $SL_2(\mathbb{Z})$-reduced point of $\left(\lfloor\sqrt{p}\rfloor + iy\right)/p$.

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