Faster Computation of Witt Vectors Ring Laws

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Constructing $\mathbb{R}$

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And its completion $\mathbb{R}$:
$\sqrt{2} = 1.4142135623\ldots$
$\pi = 3.1415926535897\ldots$

Constructing $\mathbb{R}$ in base-7

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From which we made $\mathbb{Q}$:
$\frac{3}{2} = 1.3333333333333\dots$
$\frac{1}{10} = 0.1$

And its completion $\mathbb{R}$:
$\sqrt{2} = 1.26203454521\dots$
$\pi = 3.0663651432\dots$

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Similarly, for $\mathbb{Q}$:
$\frac{3}{2} = \dots 33333335$
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We now fix a prime number p. The ring $\mathbb{Z}_p$ of p-adic integers is the set of every number in $\mathbb{Q}_p$ « without digits afters the point ».

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There is a surjective morphism $\pi\colon \mathbb{Z}_p \to \mathbb{F}_p$ of rings, which to a p-adic integer associates its rightmost digit.

$$\ldots 12345612345600123 \mapsto 3$$

$$12 \mapsto 2$$

Hensel's lemma

Hensel's lemma (1904)

Let $P \in \mathbb{Z}_p[X]$. If there is $\alpha \in \mathbb{F}_p$ such that $P(\alpha) = 0$ and $P'(\alpha) \neq 0$, then there is $x \in \mathbb{Z}_p$ such that $P(x) = 0$ and that $\pi(x) = \alpha$.

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So there are two square roots of 2 in $\mathbb{Z}_7$:
$\ldots 421216213$
$\ldots 245450454$

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Notice that $\mathbb{Z}_p / p^{\mathtt{n}} \mathbb{Z}_p \cong \mathbb{Z} / p^{\mathtt{n}} \mathbb{Z}$.

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We still lose precision when dividing by $\ldots 0010 \times p^0$.

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We do not lose precision any longer after dividing by this number! But now, there is overflow and underflow with e.

Unramified extensions

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If one takes a polynomial $P(X) \in \mathbb{Z}_p[X]$ reducing to an irreducible polynomial in $\mathbb{F}_p$, it defines an extension of finite fields $\mathbb{F}_p \to \mathbb{F}_q \cong \mathbb{F}_p[X]/\langle P(X) \rangle$.

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Philosophically, we have allowed division by p in finite fields!

Some strange ring laws

There is a bijection between $\mathbb{Z}_p$ and ${\mathbb{F}_p}^{\mathbb{N}}$ given by looking at the sequence of the digits of a p-adic integer.

$$\begin{aligned} \ldots 421216213 &\longleftrightarrow (3, 1, 2, 6, 1, 2, 1, 2, 4, \ldots) \\ 54 &\longleftrightarrow (4, 5, 0, 0, 0, 0, 0, 0, 0, \ldots) \end{aligned}$$

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This is something a mathematician has to generalise!

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\begin{aligned}
W_0 &= X_0 \\
W_1 &= {X_0}^p + pX_1 \\
W_2 &= {X_0}^{p^2} + p{X_1}^p + p^2 X_2 \\
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Think of them as truncations of p-adic expansions (for instance $201 = 1^{p^2} + p \times 0^p + p^2 \times 2$ in $\mathbb{Z}_p$).

Witt vectors polynomials

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We find:

$$S_1 = X_1 + Y_1 - \sum_{i=1}^{p-1} \frac{\binom{i}{p}}{p} X_0{}^i Y_0{}^{p-i}.$$

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We define:
$(x_0, x_1, x_2, \dots) + (y_0, y_1, y_2, \dots) := (S_0(x_0, y_0), S_1(x_0, x_1, y_0, y_1), S_2(\dots), \dots)$.

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2. to compute p-adic cohomology (Davis–Langer–Zink, 2011);
3. to try to break NTRU (Silverman–Smart–Vercauteren, 2005)...

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I have implemented in SageMath a new algorithm when the coefficient ring is a polynomial ring over $\mathbb{F}_q$.

Similarly as with absolute capping, one only compute with the first `n` coefficients of Witt vectors in practice.

It is expensive to compute with Witt vectors using the naive algorithm: when $p = 31$ the polynomial S_2 has 152994 coefficients!

Only Finotti's algorithm (2014) used another approach when the coefficient ring has characteristic p. It is much faster.

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[1] Vite.

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There is a morphism of rings $F \colon W_p(A) \to W_p(A)$ such that for $a \in A$ in characteristic p:

$$p[a] = F \circ V([a]) = V \circ F([a]),$$
$$[a]^p = [a^p] = F([a]).$$

The main lemma

Lemma (M--B)

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We conclude because $p = V \circ F$ and
$V^n(W(A))V^m(W(A)) \subset V^{n+m}(W(A))$. $\square$

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Repeat!

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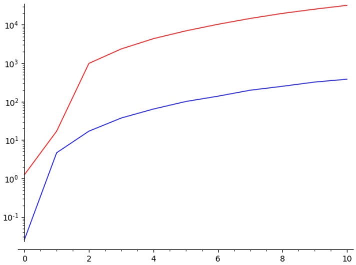
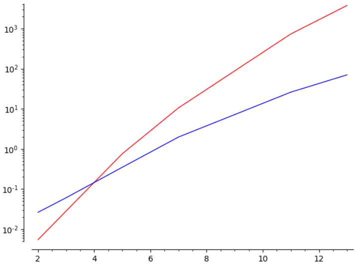


$\mathtt{n} = 5$, $p = 7$



$d = 5$, $\mathtt{n} = 4$

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Using Finotti's algorithm, Groves gave in 2023 some examples for $p > 2$. He noticed that singular curves gave the best results: one example over $\mathbb{F}_3$ had length 18, 3^6 codewords and minimum weight 8. (the best one has weight 9)

Thank you for listening!