# Another L makes it better? Lagrange meets LLL and may improve BKZ pre-processing

**Sébastien BALNY**, Claire DELAPLACE, Gilles DEQUEN

# Context

**Today**

factorization or discrete logarithms

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New enemy

Quantum computer

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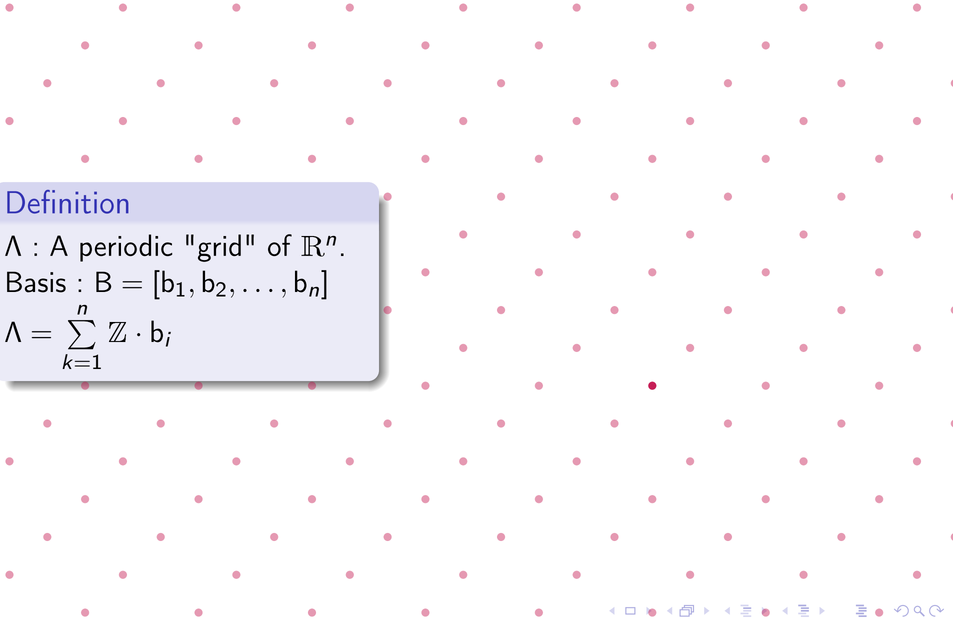
$\boxed{\text{LATTICE BASED CRYPTOGRAPHY}}$

# What is a lattice ?



**Definition**

$\Lambda$ : A periodic "grid" of $\mathbb{R}^n$.

Basis : $B = [b_1, b_2, \ldots, b_n]$

$$\Lambda = \sum_{k=1}^{n} \mathbb{Z} \cdot b_i$$

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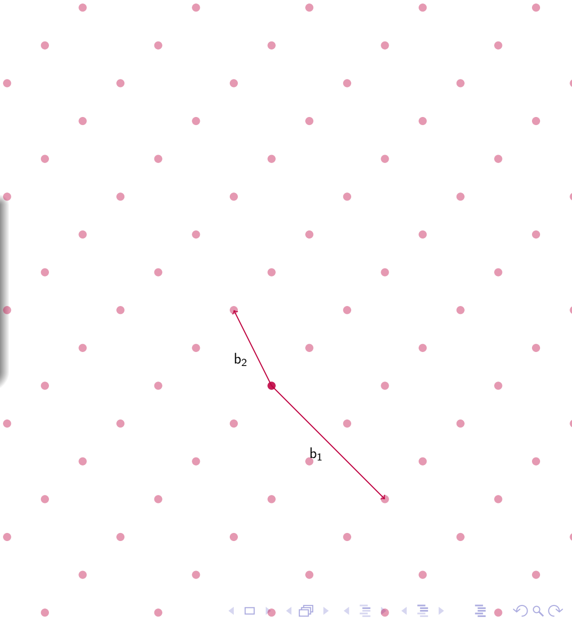

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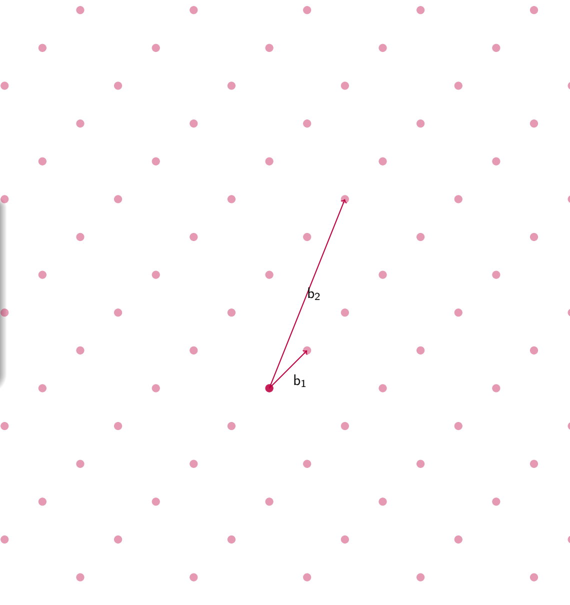

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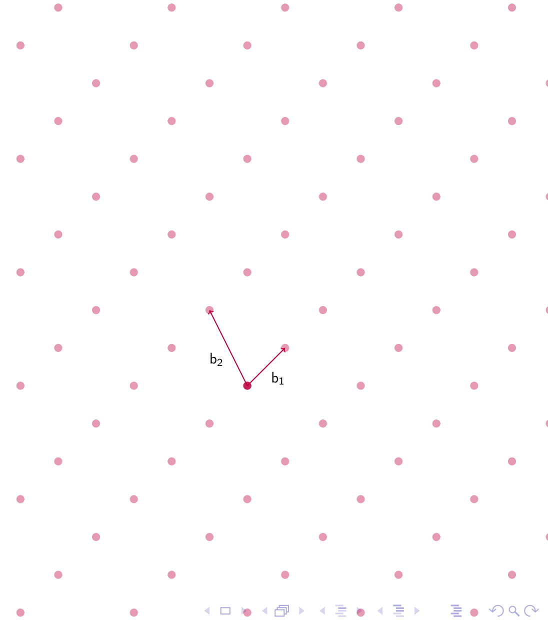

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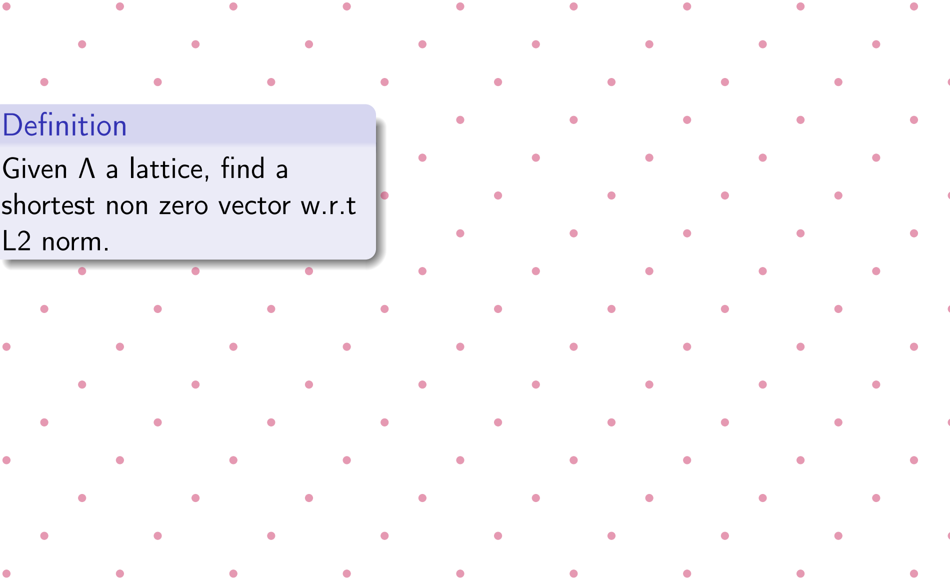
**Definition**

Given $\Lambda$ a lattice, find a shortest non zero vector w.r.t L2 norm.



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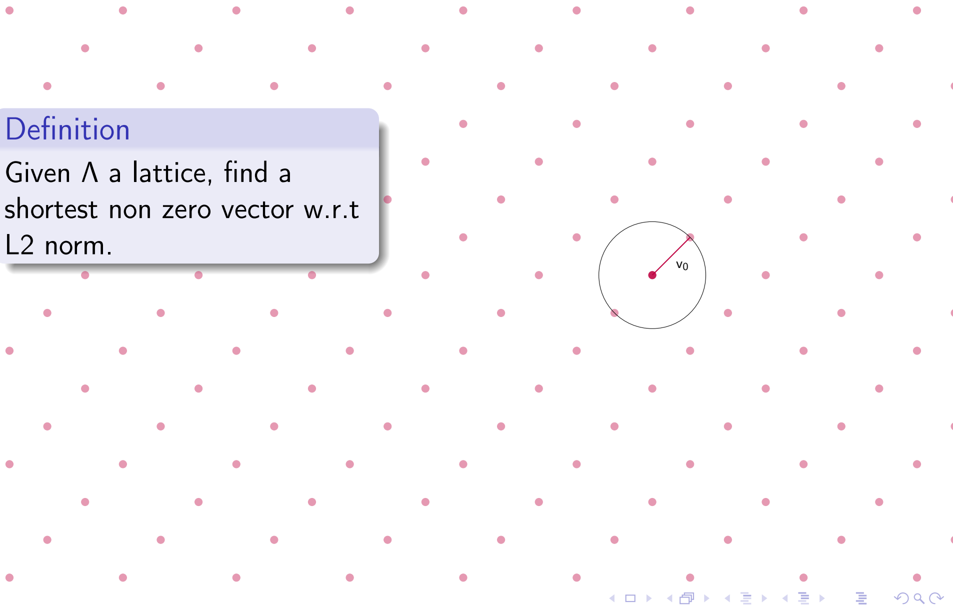


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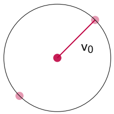


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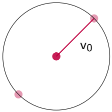
$$\boxed{\|v_0\| = \lambda_1(\Lambda)}$$



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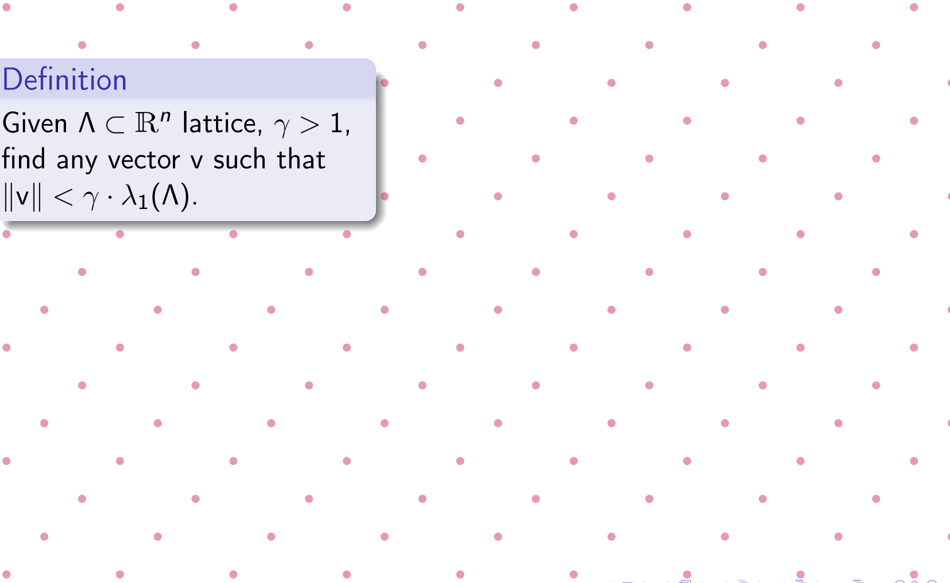

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Given $\Lambda \subset \mathbb{R}^n$ lattice, $\gamma > 1$, find any vector v such that $\|v\| < \gamma \cdot \lambda_1(\Lambda)$.

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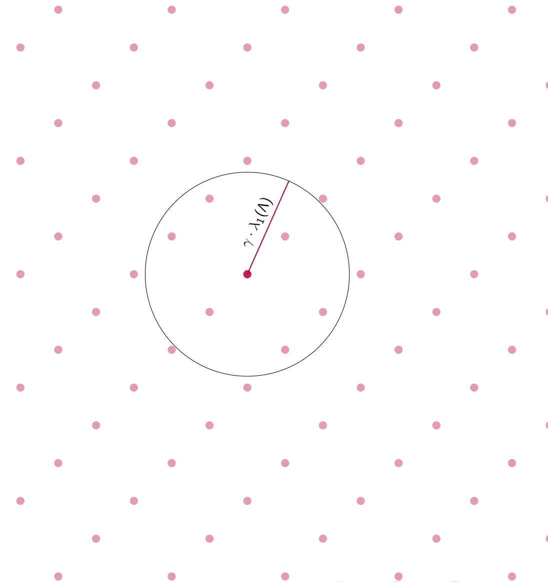

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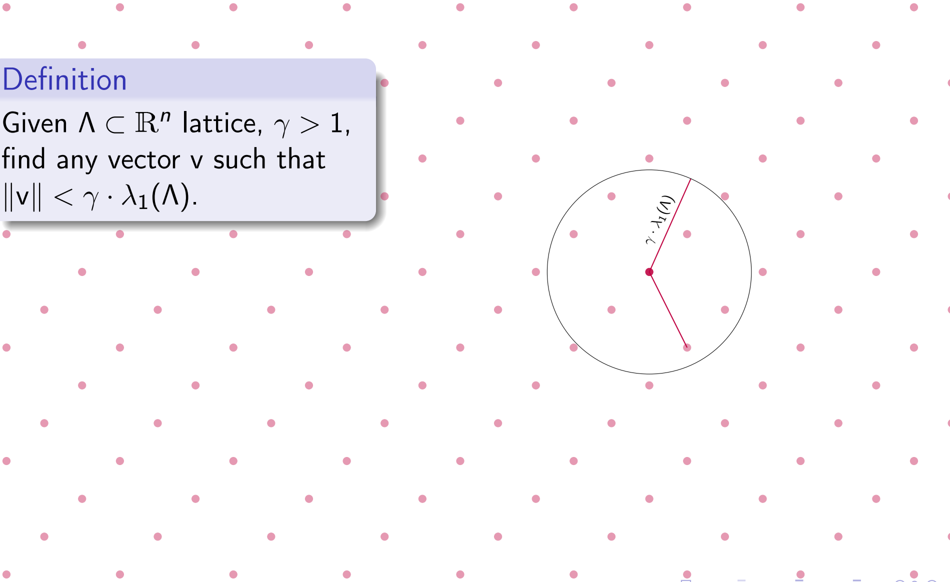

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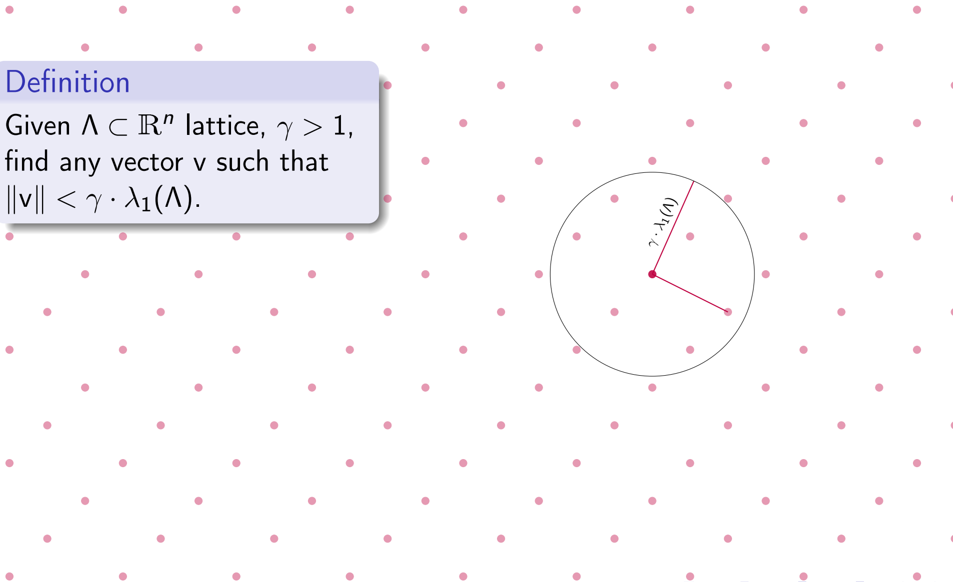

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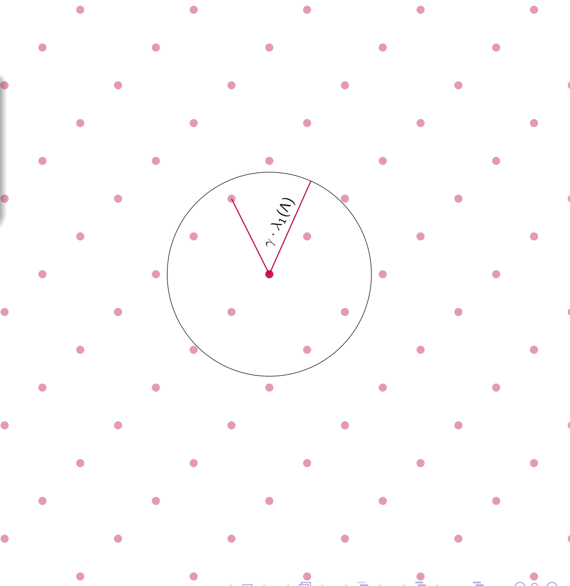

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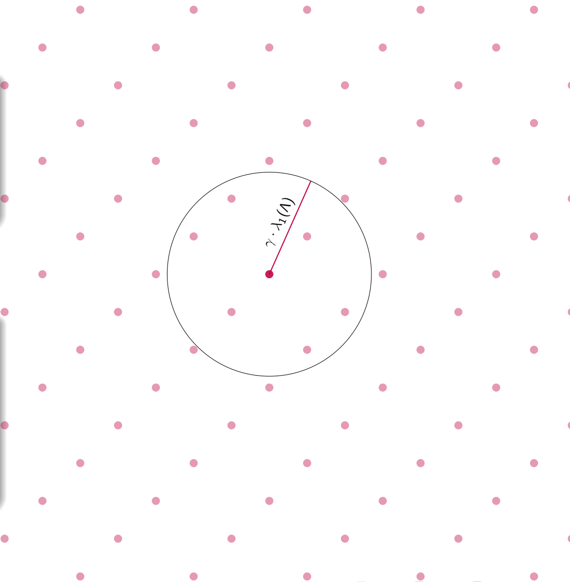
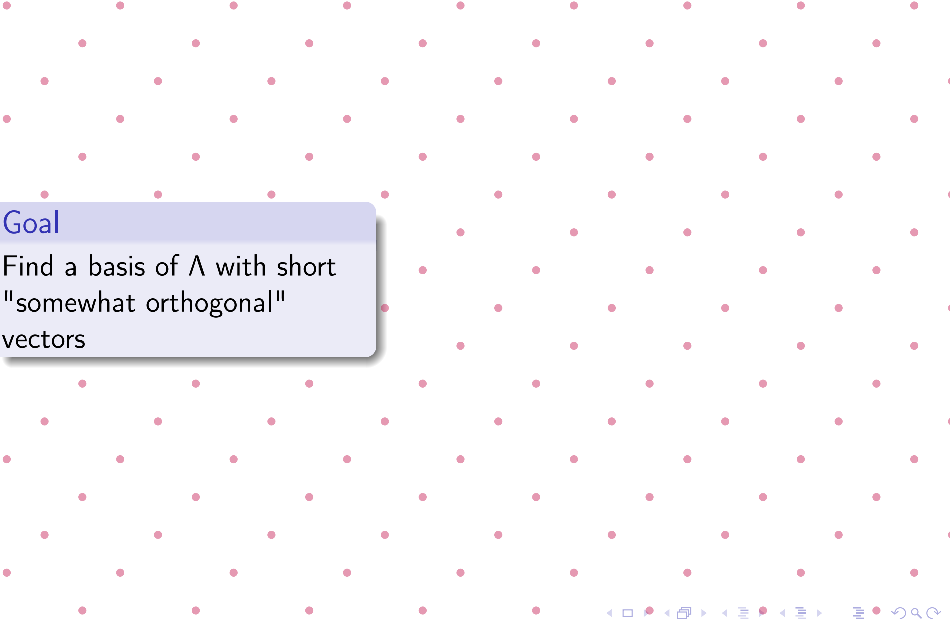
- $\gamma = \Theta(2^n) \Rightarrow \mathrm{poly}(n)$ algo [LLL1982]
- $\gamma = \Omega(n/\log(n)) \Rightarrow$ NP-hard [GG2000]

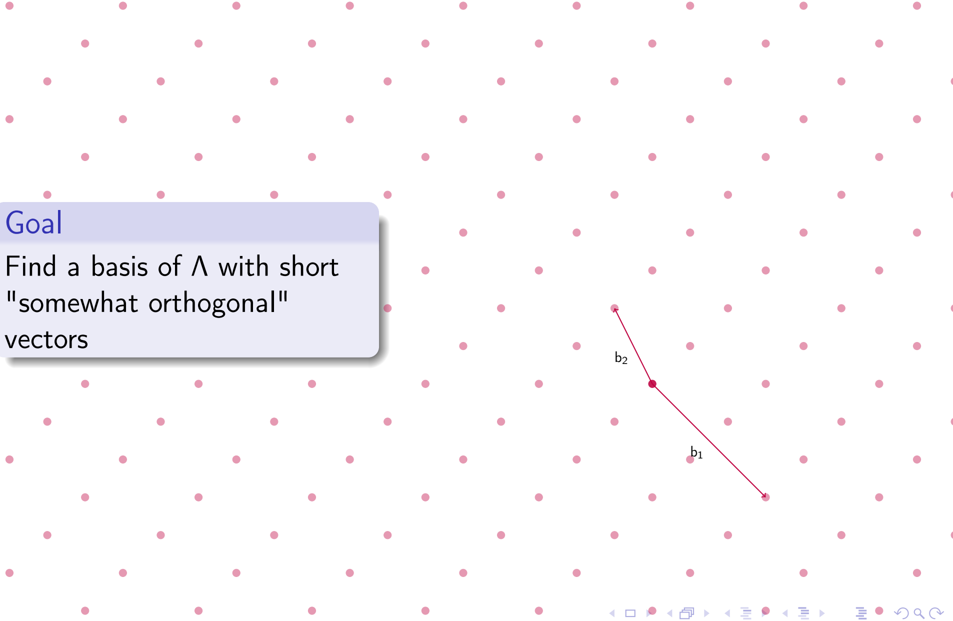




**Goal**

Find a basis of $\Lambda$ with short "somewhat orthogonal" vectors

# Lattice reduction

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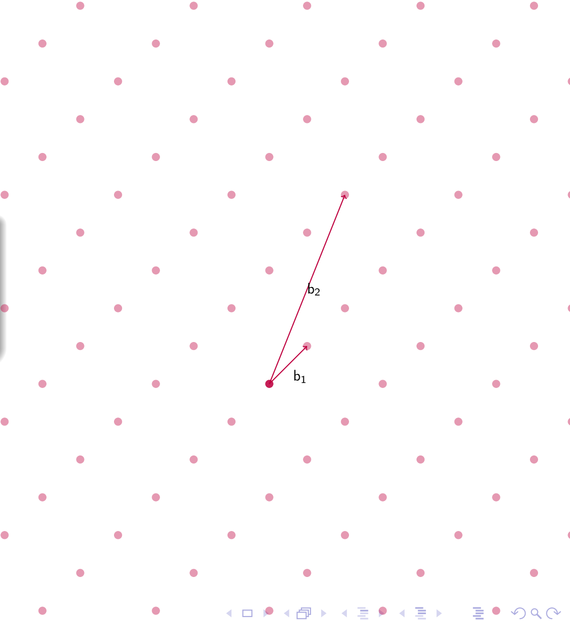

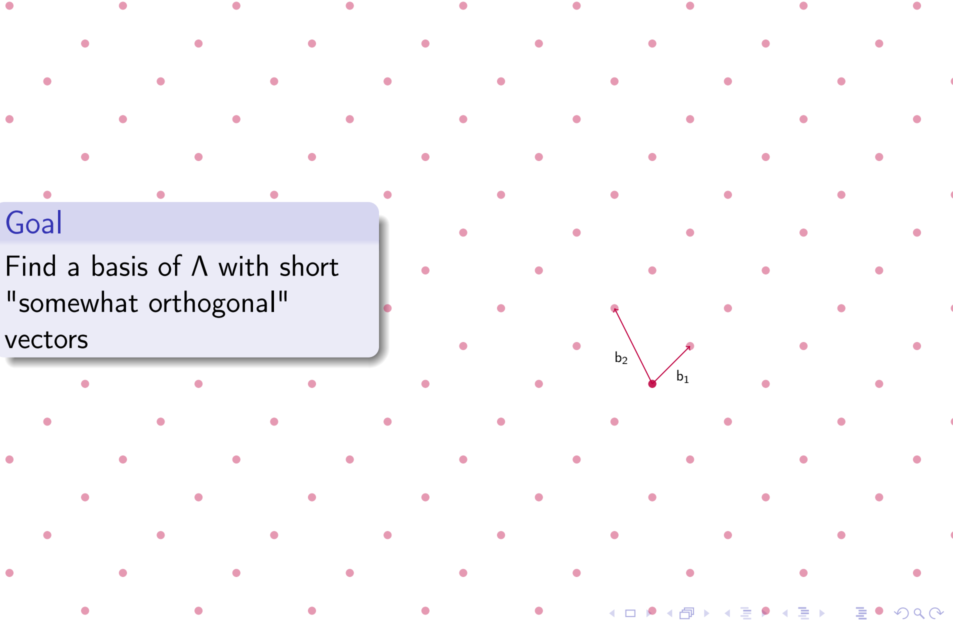


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## Block-Korkine-Zolotarev (BKZ-$\beta$) [Schnorr1987]

$\beta$ offers a trade-off between quality and efficiency.

- Runtime : after a number of tours at most $\Theta(n^2 \log n/\beta^2)$ the first basis vector of BKZ is short. Tour complexity : $2^{\mathcal{O}(\beta^2)}$. [LN2020]
- memory : $n^{\mathcal{O}(1)}$

# dimension 2 : The Lagrange algorithm

Input : Given a basis $B = [b_1, b_2]$.
Output : A reduced basis $[b_1, b_2]$.

While $|{b_2}^t b_1| > \min(\|b_1\|^2, \|b_2\|^2)/2$

1. $b_2 \leftarrow b_2 - \lfloor \mu_{2,1} \rceil b_1$
   with $\mu_{2,1} = \dfrac{|{b_2}^t b_1|}{\|b_1\|^2}$
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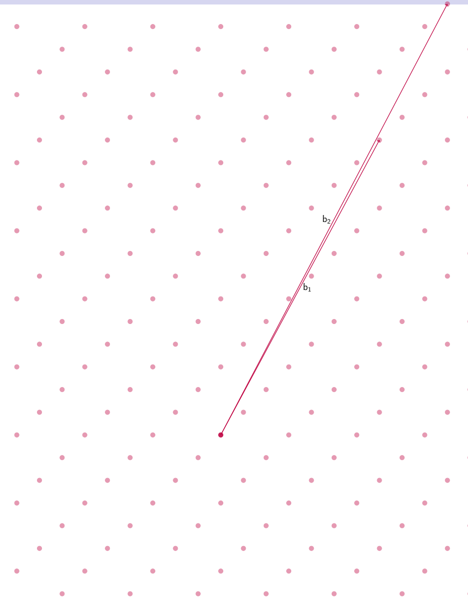

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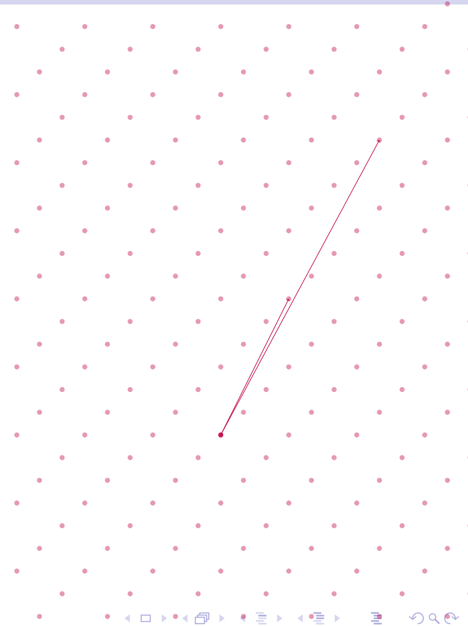

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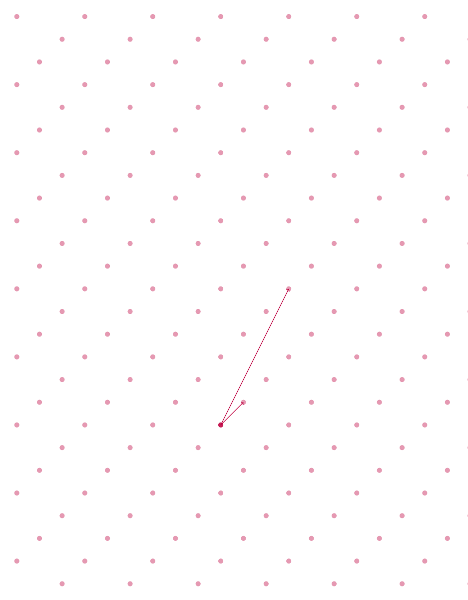

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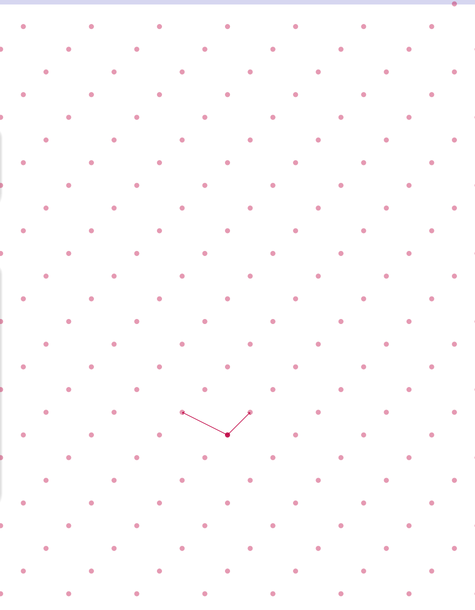

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$$\boxed{B \text{ LLL-reduced} \quad \not\Rightarrow \quad B \text{ L-reduced.}}$$

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A new algorithm inspired by **LLL** and **Lagrange**.

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- Test Lattices : Darmstadt SVP Challenge generator

  `https://www.latticechallenge.org/svp-challenge/`

  1000 lattices per dim : from 40 to 200, step of 10 (in almost cases)
- Comparison between our result $v_0$ and $\tilde{\lambda}_1 \approx \lambda_1(\Lambda)$ :

$$\boxed{\text{approx factor} = \frac{\|v_0\|}{\tilde{\lambda}_1}}$$

- Python implementation using FpyLLL library on MatriCS HPC Platform : `https://www.matrics.u-picardie.fr/`

# Our new L4 (Lagrange-LLL) algorithm

Input : A LLL-reduced basis B.
Output : A better LLL-reduced basis $B'$ with $\|b_1'\| \leq \|b_1\|$.

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2. $\mathrm{B} = \mathrm{LLLReduce}(S)$ ;
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# Construction of $S$

**Sample L4**

1. $S \leftarrow \mathrm{B}$ ;
2. Repeat $\alpha_1 n$ times :
   a. $\mathrm{u} = \mathrm{RandomChoose}(S)$ ;
   b. Repeat $\alpha_2 n$ times :
      i. $\mathrm{v} = \mathrm{RandomChoose}(S)$ ;
      ii. If $0 < \|\mathrm{u} \pm \mathrm{v}\| \leq \max(\|\mathrm{u}\|; \|\mathrm{v}\|)\|$ do
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$\alpha_1 = 1$ and $\alpha_2 = 1/2 \quad \Longrightarrow \quad$ faster than LLL.

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**Complexity**

Time complexity : $\mathcal{O}(k \times C_{LLL})$

$k$ number of calls to LLL

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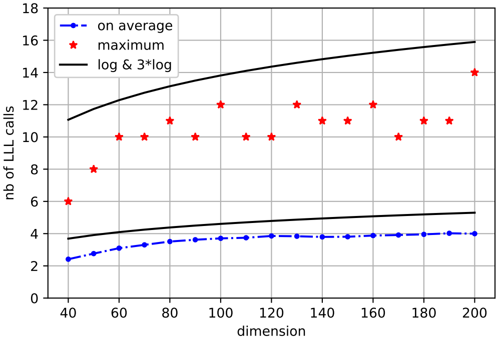
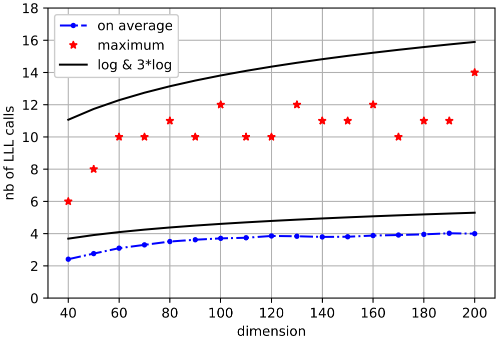



1000 tests/dim

# number of LLL calls

## Conjecture

$$k = \mathcal{O}(\log(n))$$

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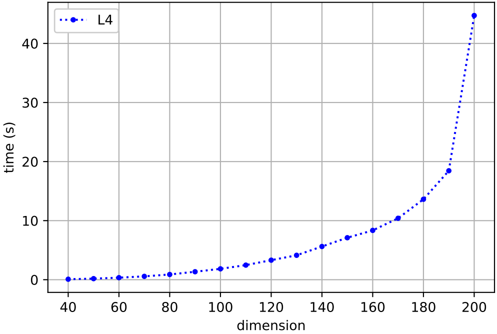


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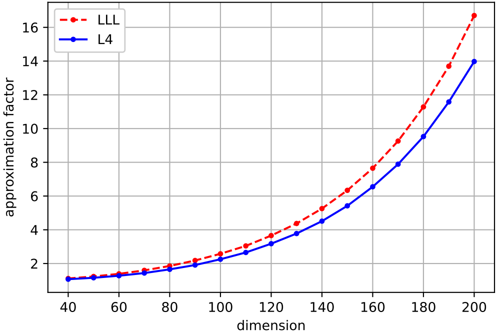
# L4 : Norm of the First Vector

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# Randomization

## Equivalent bases

$\Lambda = \mathcal{L}(\mathrm{B}_1) = \mathcal{L}(\mathrm{B}_2) \iff \mathrm{B}_2 = \mathrm{B}_1 \times U$ , for some $U$ unimodular.

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## Randomization

Input : A basis $B_1$.

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1. Generate randomly $U$ such that $\det U = \pm 1$;
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- L4-Max $k$ : stop if no improvement after $k$ randomizations.

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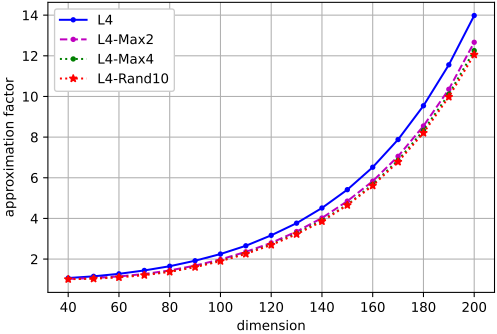


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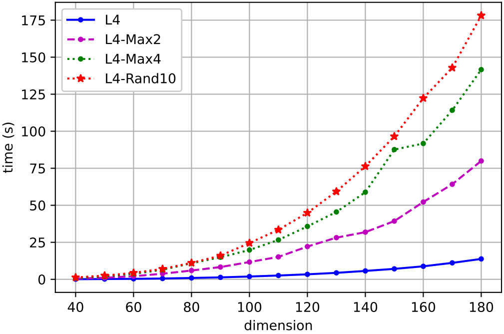
# Randomization : Experimental runtime

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# Darmstadt SVP Challenge : $\|b_1\| \leq 1.05\tilde{\lambda}_1$ ?

| Dimension | LLL | L4 | L4-Max2 | L4-Max4 | L4-Rand10 |
|---|---|---|---|---|---|
| 40 | 161 | 355 | 760 | 842 | 915 |
| 50 | 9 | 64 | 318 | 544 | 626 |
| 60 | 0 | 4 | 34 | 103 | 93 |
| 70 | 0 | 0 | 1 | 2 | 4 |
| 80 | 0 | 0 | 0 | 0 | 0 |
| 90 | 0 | 0 | 0 | 0 | 0 |

$\vdots$

1000 tests/dim

# L4 VS BKZ

## L4 VS BKZ-12

$\rightarrow$ similar runtime

$\rightarrow$ worst approximation factor

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## L4 VS BKZ-12

$\rightarrow$ similar runtime

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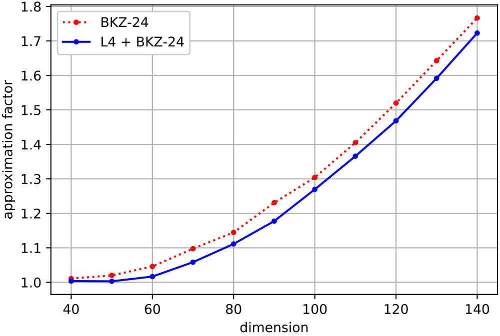
## Idea 1 : Using BKZ instead of LLL in L4

Not working : Too many L-reduced basis vectors

## Idea 2 : Using L4 instead of LLL in the pre-computation of BKZ

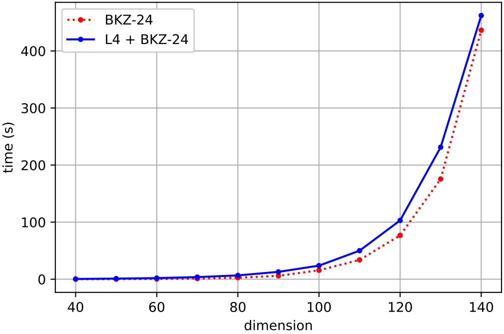
**Better results ! ! !**

# Average approximation factor




100 tests/dim

# Average runtime




100 tests/dim

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## L4+BKZ-24

- approx factor 3% better on average.
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## L4+BKZ-24 VS BKZ-24

- For all dim, there are some cases where **everything is better**.

# Conclusion

- $\rightarrow$ A new algorithm inspired by LLL and Lagrange reduction
- $\rightarrow$ Improve the approximation factor for SVP when used as pre-processing of BKZ
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# Thank you !

https://zenodo.org/records/13847623