On dual attacks against the Learning With Errors problem

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Based on joint works with

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Learning with errors (LWE)

Let $n = 4$, $m = 6$ and $q = 17$.

secret

$$\underset{A \in \mathbb{Z}_q^{m \times n}}{\begin{bmatrix} 14 & 12 & 2 & 5 \\ 5 & 3 & 1 & 7 \\ 14 & 7 & 2 & 5 \\ 0 & 9 & 8 & 4 \\ 8 & 11 & 5 & 12 \\ 5 & 1 & 3 & 14 \end{bmatrix}} \times \underset{s \in \mathbb{Z}_q^{n}}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} = \underset{b \in \mathbb{Z}_q^{m}}{\begin{bmatrix} 11 \\ 5 \\ 14 \\ 6 \\ 12 \\ 13 \end{bmatrix}}$$

Given A and b, find s.

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$$

Given A and b, find s.

$\rightsquigarrow$ Very easy (e.g. Gaussian elimination) and in polynomial time

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random	secret	noise	
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Given A and b, find s.

$\rightsquigarrow$ Suspected hard problem, even for quantum algorithms

Learning with errors (LWE)

Let $n, m, q \in \mathbb{Z}$ and χ_e, χ_s two distributions over $\mathbb{Z}_q$.

LWE$(n, m, q, \chi_e, \chi_s)$: probability distribution on $\mathbb{Z}_q^{m\times n} \times \mathbb{Z}_q^m$

- sample $A \leftarrow U(\mathbb{Z}_q^{m\times n})$
- sample $s \leftarrow \chi_s^n$
- sample $e \leftarrow \chi_e^m$
- output $(A, As + e)$.

Intuition: $As + e$ is very close to a uniform distribution.

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Search LWE problem: given $(A, b) \leftarrow$ LWE$(n, m, q, \chi_e, \chi_s)$, recover s.

Decision LWE problem:
distinguish LWE$(n, m, q, \chi_e, \chi_s)$ from $U(\mathbb{Z}_q^{m \times n} \times \mathbb{Z}_q^m)$.

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Lemma: Search LWE is easy if and only if decision LWE is easy.

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Secret distributions χ_s:

- originally uniform in $\mathbb{Z}_q$, now some distribution of small deviation σ_s (e.g. discrete Gaussian/centered Binormial, $\{-1, 0, 1\}$ whp)
- Fact: small secret is as hard as uniform secret
- small secret allows more efficient schemes

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Noise distributions χ_e:

- usually discrete Gaussian/centered Binormial of deviation σ_e
- most schemes (Kyber/Saber/...): σ_e small (≈ 1)

LWE: security and attacks

LWE is fundamental to lattice-based cryptography:

- several lattice-based NIST PQC candidates rely on LWE
- extensive literature
- all evidence points to resistance against quantum attacks

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Two types of attacks:

- Primal attacks:
  - more efficient in most cases
  - no quantum speed-up known (besides BKZ)
- Dual attacks:
  - originally less efficient, now catching up
  - no quantum speed-up known (besides BKZ) up to now

Contributions:

- first quantum speed-up on dual attacks
- improvement on dual attacks using ideas from codes

Search to distinguish

Very naive attack:

$$\overset{A}{\begin{bmatrix} 8 & 9 & 10 & 12 \\ 5 & 3 & 11 & 3 \\ 0 & 15 & 10 & 4 \\ 15 & 9 & 16 & 15 \\ 1 & 2 & 10 & 8 \\ 11 & 16 & 13 & 9 \end{bmatrix}} \times \overset{s}{\begin{bmatrix} \; \\ \; \\ \; \\ \; \end{bmatrix}} + \overset{e}{\begin{bmatrix} \; \\ \; \\ \; \\ \; \\ \; \\ \; \end{bmatrix}} = \overset{b}{\begin{bmatrix} 8 \\ 3 \\ 11 \\ 15 \\ 2 \\ 15 \end{bmatrix}}$$

Attack:

- get (A, b)

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Very naive attack: guess secret $\tilde{s}$

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Attack:

- get (A, b)
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$$b' = e$$

follows a discrete Gaussian
of small deviation

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Bad guess ($s \neq \tilde{s}$):

$$b' = e + A(s - \tilde{s})$$

follows a uniform[1] distribution (A uniform in $\mathbb{Z}_q^{m \times n}$)

[1] Technically only true for fixed s, random A and $\tilde{s}$

Uniform/Gaussian distinguisher

Given a sampler for χ^m, decide if $\chi = U(\mathbb{Z}_q)$ or $D_{\sigma,q}$ (discrete Gaussian)

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The entries are independent: given a sample from χ^m we obtain m independent samples from χ.
$\rightsquigarrow$ if m large enough, we know how to distinguish.

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Essentially optimal distingusher: let $Y = \Re(e^{2i\pi X/q})$

$$\mathbb{E}_{X \leftarrow \chi}[Y] \approx \begin{cases} 0 & \text{if } \chi = U(\mathbb{Z}_q) \\ e^{-2\left(\frac{\pi\sigma}{q}\right)^2} & \text{if } \chi = D_{\sigma,q} \end{cases}$$

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Attack:

- sample $N = \Omega\left(1/\varepsilon^2\right)$ values $x_1, \ldots, x_N$ from χ
- compute

$$S = \frac{1}{N} \sum_{j=1}^{N} \Re(e^{2i\pi x_j/q})$$

- Check if $S > \frac{1}{2} e^{-2\left(\frac{\pi\sigma}{q}\right)^2}$

The quantity $\varepsilon = e^{-2\left(\frac{\pi\sigma}{q}\right)^2}$ is called the advantage.

Very naive attack: summary

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Can do better by guessing s in decreasing order of probability[1]:

$$G(\chi_s^n) \cdot e^{4\left(\frac{\pi\sigma_e}{q}\right)^2} \leqslant (1.22\sqrt{2\pi}\sigma_s)^n \cdot e^{4\left(\frac{\pi\sigma_e}{q}\right)^2} = \text{too much}$$

where σ_s deviation of s, $G(\cdot) =$ guessing complexity

[1] The complexity is now the expected running time

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Dual attacks: provide an efficient way to only guess a part of the secret

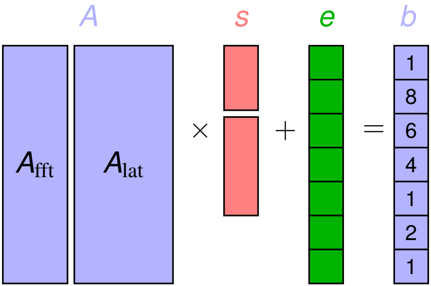
[1]The complexity is now the expected running time

Search to Decision LWE

$$\overset{A}{\begin{bmatrix} 3 & 7 & 2 & 3 & 6 \\ 4 & 1 & 5 & 8 & 4 \\ 1 & 8 & 1 & 8 & 1 \\ 5 & 2 & 5 & 6 & 0 \\ 2 & 1 & 6 & 3 & 0 \\ 8 & 2 & 7 & 3 & 6 \\ 5 & 5 & 6 & 6 & 2 \end{bmatrix}} \times \overset{s}{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}} + \overset{e}{\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}} = \overset{b}{\begin{bmatrix} 1 \\ 8 \\ 6 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}}$$

Search to Decision LWE

Split secret: $n = k_{\text{fft}} + k_{\text{lat}}$



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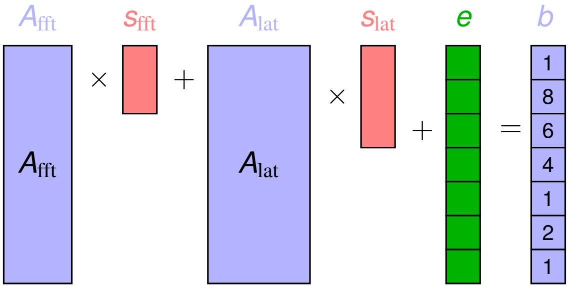

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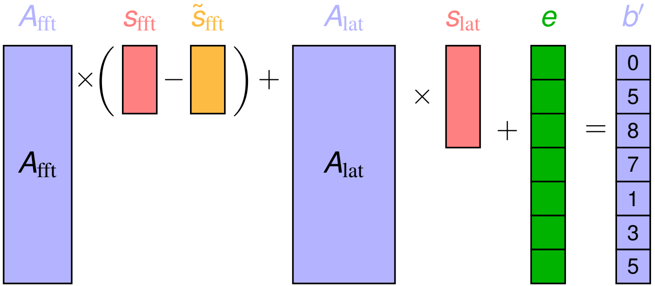
Split secret: $n = k_{\text{fft}} + k_{\text{lat}}$, guess $\tilde{s}_{\text{fft}}$, output $(A_{\text{lat}}, b' = b - A_{\text{fft}}\tilde{s}_{\text{fft}})$



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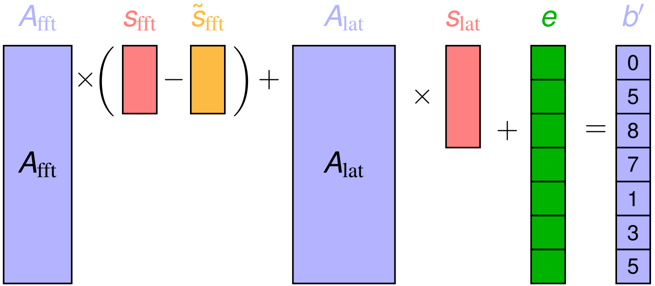

Good guess ($s_{\text{fft}} = \tilde{s}_{\text{fft}}$):

$$b' = A_{\text{lat}} s_{\text{lat}} + e$$

so (A_{lat}, b') follows an LWE distribution

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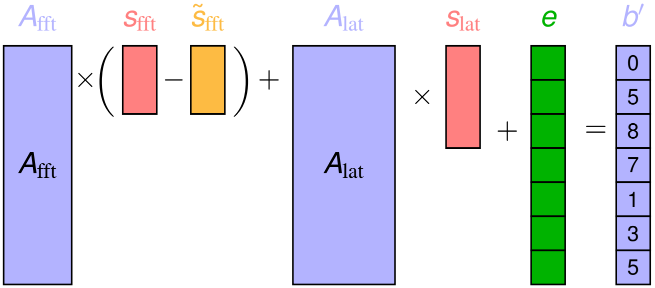

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Bad guess ($s_{\mathrm{fft}} \neq \tilde{s}_{\mathrm{fft}}$):

$$b' = A_{\mathrm{fft}}(s_{\mathrm{fft}} - \tilde{s}_{\mathrm{fft}}) + \cdots$$

so (A_{lat}, b') follows a uniform distribution (A_{fft} uniform)

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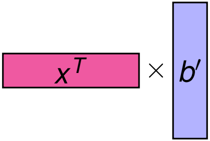

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When $\chi =$ Uniform:

$$x^T b'$$

follows a uniform distribution (b' uniform, independent from A_{lat})

Dual attack: naive complexity

Naive dual attack:

- split secret $n = k_{\text{fft}} + k_{\text{lat}}$
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- e approx Gaussian deviation σ_e
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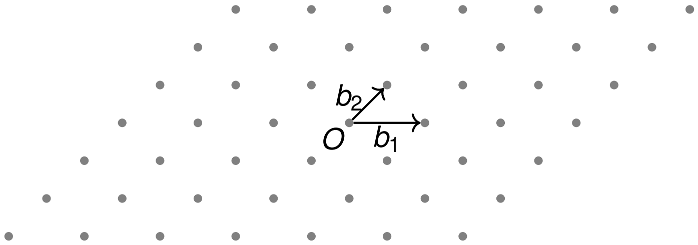
$\rightsquigarrow$ we want x to be short $\rightsquigarrow$ lattice reduction

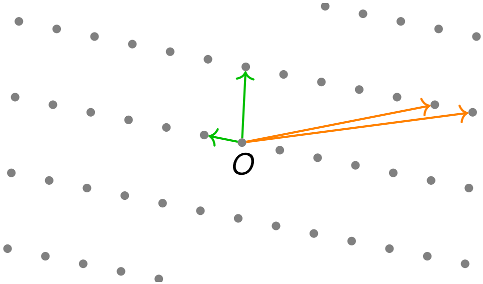
What is a (Euclidean) lattice?

Definition

$\mathcal{L}(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n) = \left\{\sum_{i=1}^{n} x_i \boldsymbol{b}_i : x_i \in \mathbb{Z}\right\}$ where $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ is a basis of $\mathbb{R}^n$.



Lattice-based cryptography: fundamental idea



- good basis: private information, makes problem easy
- bad basis: public information, makes problem hard

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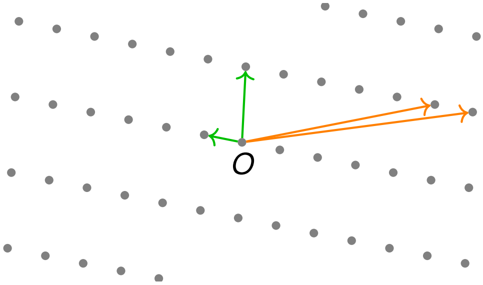

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Basis reduction: transform a bad basis into a good one
Main tool: BKZ algorithm and its variants

Requires to solve the (approx-)SVP problem in smaller dimensions.

An important optimization

We are chaining two reductions:

- $b' = b - A_{\text{fft}} \tilde{s}_{\text{fft}}$ comes from search to decision reduction
- $x_1, \ldots, x_N$ is a list of dual vectors
- $\alpha_j = {x_j}^T b'$ comes from uniform/LWE to uniform/Gaussian red.

To distinguish between unidimensional uniform/Gaussian, we compute

$$F(\tilde{s}_{\text{fft}}) = \sum_{j=1}^{N} e^{\frac{2i\pi}{q}\alpha_j}$$

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$$F(\tilde{s}_{\text{fft}}) = \sum_{j=1}^{N} e^{\frac{2i\pi}{q}\alpha_j} = \sum_{j=1}^{N} e^{\frac{2i\pi}{q}{x_j}^T(b - A_{\text{fft}}\tilde{s}_{\text{fft}})} = \sum_{j=1}^{N} e^{\frac{2i\pi}{q}{x_j}^T b} \cdot e^{-\frac{2i\pi}{q}{x_j}^T A_{\text{fft}}\tilde{s}_{\text{fft}}}$$

and we want to find $\tilde{s}_{\text{fft}}$ such that $\Re(F(\tilde{s}_{\text{fft}})) > \text{threshold}$

FFT search with threshold

Problem: given $(x_1, w_1), \dots, (x_N, w_N) \in \mathbb{Z}_q^{k_{\text{fft}}} \times \mathbb{C}$ with N large and $\delta > 0$

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Complexity:

$$\text{array filling time} + \text{FFT time} + \text{search time} = \tilde{O}(N + q^{k_{\text{fft}}})$$

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- check if any amplitude in the superposition is above the threshold ▶ extremely expensive?

Open question: can this approach be made efficient?

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- we can do better with a QRAM

Theorem (Simplified)

There is a quantum algorithm that computes $F(s) \pm \varepsilon$ given oracle access by making $O(1/\varepsilon)$ queries to $\mathcal{O}_X$:

$$\mathcal{O}_X : |j\rangle\,|0\rangle \to |j\rangle\,\left|x_j\right\rangle .$$

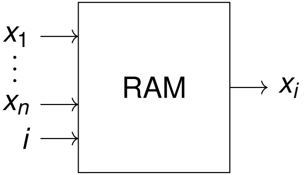
How can we build such an oracle? $\rightsquigarrow$ QRAM

Interlude: quantum memory models

classical access

quantum access

classical data



standard

quantum data

Assumption: $O(1)$ time cost

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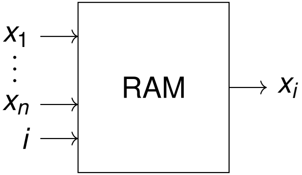

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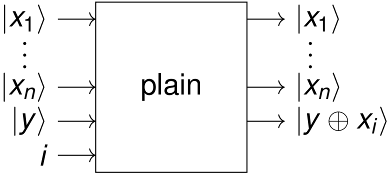

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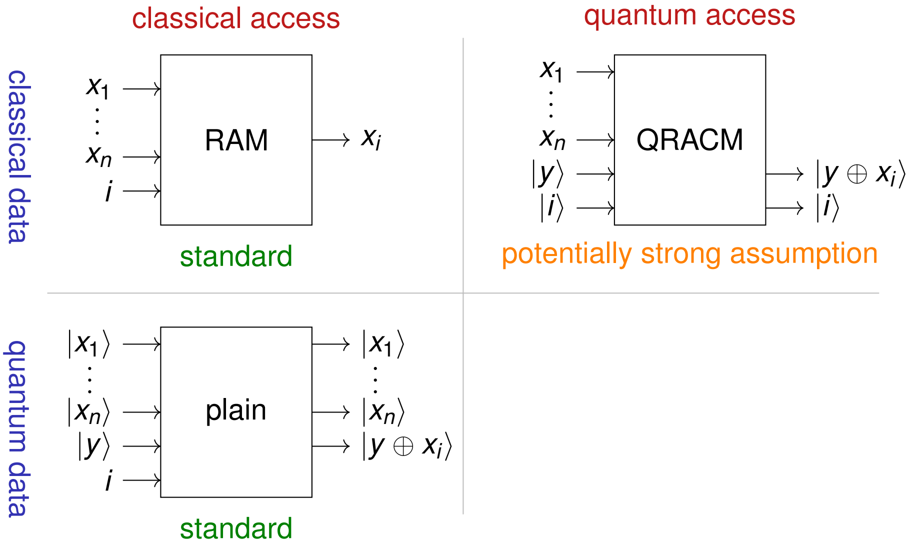

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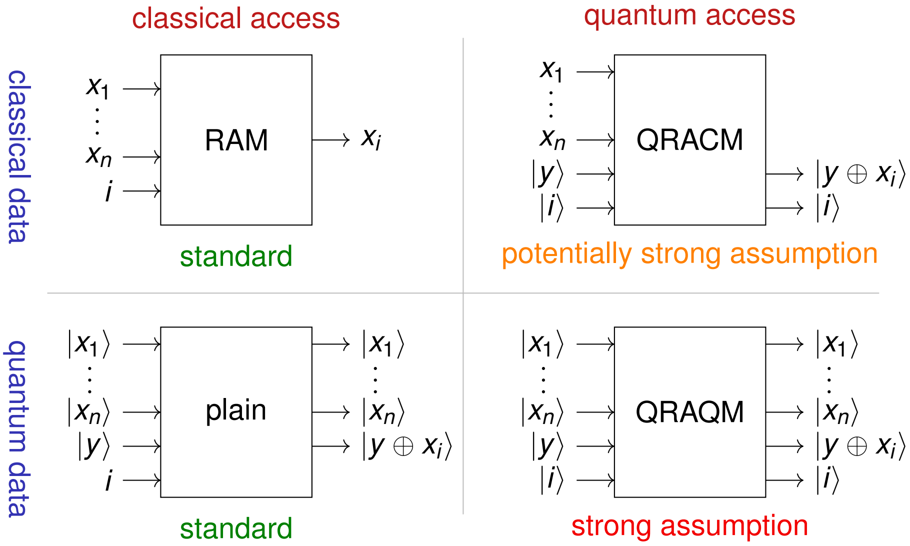

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FFT search with threshold (quantum)

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Quantum complexity	Classical complexity
$O(\sqrt{q^{k_{\text{fft}}} \cdot N})$	$O(q^{k_{\text{fft}}} + N)$

- quantum never worse than classical
- gain when $N \ll q^{k_{\text{fft}}}$ or $N \gg q^{k_{\text{fft}}}$

Dual attack: summary

- split secret $n = k_{\text{fft}} + k_{\text{lat}}$
- compute many dual vectors x
- find $\tilde{s}_{\text{fft}}$ using FFT/quantum mean estimation

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$$L = \left\{ x \in \mathbb{Z}^m : x^T A_{\text{lat}} = 0 \bmod q \right\}$$

Complexity estimate:

$$q^{k_{\text{fft}}} + e^{4\left(\frac{\pi \|x\| \sigma_e}{q}\right)^2} + T_{\text{BKZ}}$$

Classical

$$\sqrt{q^{k_{\text{fft}}} e^{4\left(\frac{\pi \|x\| \sigma_e}{q}\right)^2}} + T_{\text{BKZ}}$$

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Quantum with QRAM

- BKZ trade-off: short $x \rightsquigarrow$ more expensive algorithm
- best dual attack parameters ($k_{\mathrm{fft}}, \ldots$) found by optimization

Advanced dual attacks

Modulus switching: only guess part of secret modulo p ($p \ll q$)

- reduce guessing complexity
- increase distinguishing cost due to modulo remainders
- makes reduced secret dense

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BKZ with sieving

- obtain many dual vectors at once
- reducing the number of BKZ reductions

Hybrid dual attack

Combine enumeration with dual attack:

- enumerate $s_{\text{enum}} \in \mathbb{Z}_q^{k_{\text{enum}}}$ sampled from $\chi_s^{k_{\text{enum}}}$
  - enumerate all $s_{\text{fft}} \in \mathbb{Z}_p^{k_{\text{fft}}}$ uniform in $\mathbb{Z}_p^{k_{\text{fft}}}$
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- guessing complexity: try s_{enum} in decreasing order of probability
- FFT: compute all DFT-sums in one go with a FFT
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Gain: reduce $k_{\mathrm{lat}} \rightsquigarrow$ decrease BKZ cost

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Classical:

$$G(\chi_s^{k_{\text{enum}}}) \cdot \left(p^{k_{\text{fft}}} + e^{4\left(\frac{\pi \|x\| \sigma_e}{q}\right)^2} \right) + T_{\text{BKZ}}$$

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Quantum with QRAM:

$$G^{qc}(\chi_s^{k_{\text{enum}}}) \cdot \sqrt{p^{k_{\text{fft}}} \cdot e^{4\left(\frac{\pi \|x\| \sigma_e}{q}\right)^2}} + T_{\text{BKZ}}$$

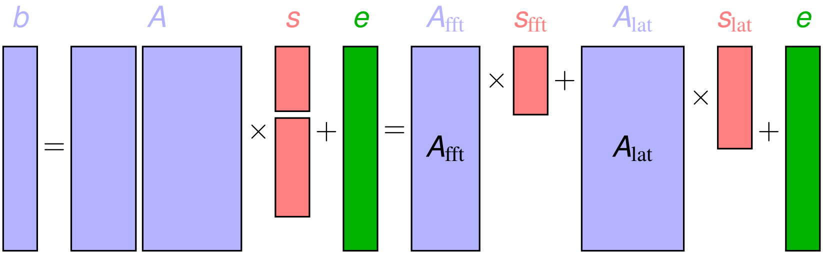
Dual attack cost estimates (logarithms to base two)

	Classical			Quantum		Our work	
Scheme	CC	CN	C0	QN	Q0	QN	Q0
Kyber 512	139.2	134.4	115.4	124.4	102.7	119.3	99.6
Kyber 768	196.1	190.6	173.7	175.3	154.6	168.2	149.8
Kyber 1024	262.4	256.1	241.8	234.5	215.0	226.0	208.5
LightSaber	138.5	133.1	113.7	122.7	101.1	118.6	98.5
Saber	201.4	195.9	179.2	179.9	159.4	175.6	155.7
FireSaber	263.5	258.2	243.8	235.9	216.7	228.3	210.7
TFHE630	118.2	113.3	93.0	105.2	83.0	102.6	81.6
TFHE1024	122.0	117.2	95.4	108.5	84.8	106.6	83.5

- QN: quantum version of CN
- Q0: quantum version of C0
- CC: classical circuit model (most detailed)
- CN: intermediate model
- C0: Core-SVP model (very pessimistic)

Recall: split secret + dual vector

Combine: split secret



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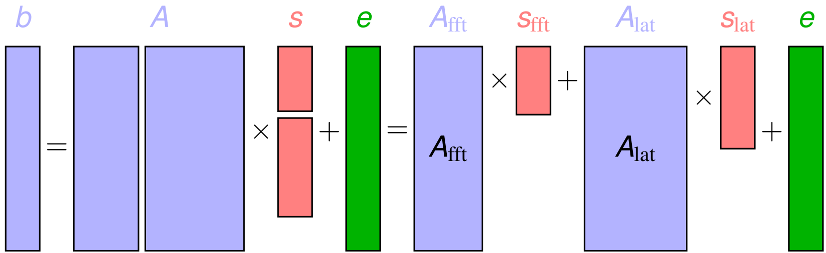
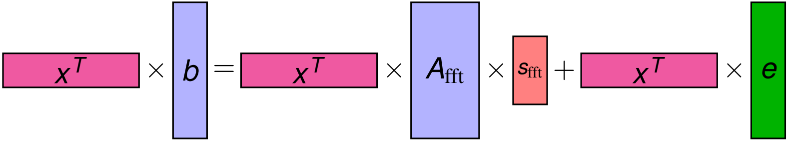


With: dual vector x such that $x^T A_{\text{lat}} = 0$



Fundamental equation of dual attack

- split secret, find (x, y) such that $x^T A_{\text{lat}} = 0$ and $y^T = x^T A_{\text{fft}}$

$$x^T \times b = y^T \times s_{\text{fft}} + x^T \times e$$

Fundamental equation of dual attack

- split secret, find (x, y) such that $x^T A_{\mathrm{lat}} = 0$ and $y^T = x^T A_{\mathrm{fft}}$
- guess secret $\tilde{s}$ and subtract

$$x^T \times b - y^T \times \tilde{s}_{\mathrm{fft}} = y^T \times \left(s_{\mathrm{fft}} - \tilde{s}_{\mathrm{fft}} \right) + x^T \times e$$

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follows a discrete Gaussian of
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Bad guess ($s_{\text{fft}} \neq \tilde{s}_{\text{fft}}$):

$$y^T(s_{\text{fft}} - \tilde{s}_{\text{fft}}) + x^T e$$

follows a uniform distribution ($y \approx$ uniform in $\mathbb{Z}_q^{k_{\text{fft}}}$)

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Bad guess ($s_{\text{fft}} \neq \tilde{s}_{\text{fft}}$):

$$y^T(s_{\text{fft}} - \tilde{s}_{\text{fft}}) + x^T e$$

follows a uniform distribution ($y \approx$ uniform in $\mathbb{Z}_q^{k_{\text{fft}}}$)

Problem: cost of distinguishing grows as $q^{k_{\text{fft}}}$

$\rightsquigarrow$ can we change to a modulo $p \ll q$ to reduce the cost?

Modulus switching from a high level

Let $p < q$, write

$$py = qu + t$$

where $u \in \mathbb{Z}_p^{k_{\mathrm{lat}}}$ and $t \in \mathbb{Z}_q^{k_{\mathrm{lat}}}$.

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$$p\, x^T \cdot b - q\, u^T \cdot \tilde{s}_{\mathrm{fft}} = q\, u^T \cdot \left(s_{\mathrm{fft}} - \tilde{s}_{\mathrm{fft}} \right) + \varepsilon$$

where $\varepsilon = t^T \cdot s_{\mathrm{fft}} + p\, x^T \cdot e$

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This is a trade-off (details omitted):

- only need to guess $s_{\text{fft}} \bmod p$: FFT over $\mathbb{Z}_p^{k_{\text{fft}}}$ instead of $\mathbb{Z}_q^{k_{\text{fft}}}$
- the error ε has increased: the number of samples increases

$$\text{from} \quad 4\left(\frac{\pi \|x\| \sigma_e}{q}\right)^2 \quad \text{to} \quad 4\left(\frac{\pi \|x\| \sigma_e}{q}\right)^2 + \frac{1}{3}\left(\frac{\pi \|s_{\text{fft}}\| q}{p}\right)^2$$

Going further: using ideas from coding theory

Everyting until this point is in the LWE report by the MATZOV group.

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Modulus switching: approximate a vector $y \in \mathbb{Z}_q^n$ by

$$y = \frac{q}{p} \cdot [\frac{p}{q} y] + \frac{q}{p}\{\frac{p}{q} y\} = \frac{q}{p} \cdot u + t$$

- $u \in \mathbb{Z}_p^n$: smaller domain (field is smaller)
- $\|t\| \leqslant \frac{q}{p}$: “small error”

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Our observation: this looks like a special case of lattice codes

$$y = Gu + t$$

- $G \in \mathbb{Z}_q^{n \times k}$: defines a code
- $u \in \mathbb{Z}_q^k$: smaller domain (dimension is smaller)
- $\|t\|$ is small (depends on G)

Applying lattice codes

Recall: find (x, y) such that $x^T A_{\text{lat}} = 0$ and $y^T = x^T A_{\text{fft}}$

$$x^T \times b = y^T \times s_{\text{fft}} + x^T \times e$$

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Choose a code $G \in \mathbb{Z}_q^{k_{\mathrm{fft}} \times k_{\mathrm{cod}}}$, decode y as

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New fundamental equation:

$$x^T \cdot b = u^T \cdot G^T \cdot s_{\mathrm{fft}} + t^T \cdot s_{\mathrm{fft}} + x^T \cdot e$$

Lattice codes: fundamental equation

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Observations:

- we directly guess s_{cod} instead of s_{fft}
- $s_{\mathrm{cod}} = G^T s_{\mathrm{fft}} \in \mathbb{Z}_q^{k_{\mathrm{cod}}}$ has **smaller dimension** $k_{\mathrm{cod}} \ll k_{\mathrm{fft}}$

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- $\varepsilon' = t^T s_{\text{fft}} + x^T e$ follows a discrete Gaussian whose **deviation** depends on $\|x\|$, $\|s_{\text{fft}}\|$, $\|e\|$ and $\|t\|$
- $\|t\|$ is **small** for a good code G

Lattice codes vs modulo switching

Lattice codes

$$x^T \cdot b = u^T \cdot s_{\text{cod}} + \varepsilon'$$

Modulus switching

$$x^T \cdot b = \left[\frac{p}{q}\, y^T\right] \cdot s_{\text{fft}} + \varepsilon$$

Lattice codes vs modulo switching

Lattice codes

$$x^T \cdot b = u^T \cdot s_{\mathrm{cod}} + \varepsilon'$$

- FFT cost: $q^{k_{\mathrm{cod}}}$
- error ε': Gaussian of stddev

$$\tau_{\mathrm{LC}}^2 = \|x\|^2 \cdot \sigma_e^2 + \|s_{\mathrm{fft}}\|^2 \cdot \frac{q^{2-2\frac{k_{\mathrm{cod}}}{k_{\mathrm{fft}}}}}{2\pi e}$$

for an asymptotically optimal code

Modulus switching

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Comparison for same FFT cost: $q^{k_{\mathrm{cod}}} = p^{k_{\mathrm{fft}}}$

$$\frac{q^{2-2\frac{k_{\mathrm{cod}}}{k_{\mathrm{fft}}}}}{2\pi e} = \frac{q}{2\pi e p} \approx \frac{q}{17p} \ll \frac{q}{12p}$$

$\rightsquigarrow$ lattice codes are always better than modulo switching!

Other important details

- FFT is more efficient for powers of two
- $q^{k_{\text{cod}}}$ has coarse granularity for big q

$\rightsquigarrow$ use modulo switching to change q to $p = 2^m$ then use lattice codes: best of both, allow more "continuous" parameter choice

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- many parameters to choose (p, k_{fft}, k_{cod}, BKZ block size, ...)
- no obvious way to choose them

$\rightsquigarrow$ search for optimal parameters with an optimisation program

Results

- CC: classical circuit model (most detailed cost)
- CN: intermediate model
- C0: “Core-SVP” cost model

Scheme	MATZOV CC	MATZOV CN	MATZOV C0	Ours CC	Ours CN	Ours C0
Kyber 512	138.5	133.7	114.8	137.8	133.0	114.0
Kyber 768	195.7	190.4	173.1	192.5	187.2	170.2
Kyber 1024	261.4	255.4	240.7	256.2	250.5	235.7
LightSaber	137.1	132.3	113.1	136.8	131.5	112.3
Saber	201.1	195.1	178.3	199.7	194.9	177.0
FireSaber	263.6	257.7	242.8	259.9	254.4	239.4

- 1 to 5 bit gain over MATZOV
- further 1 bit gain with Prange bet (not in the talk)