Cryptanalyse logique du probleme du logarithme discret sur courbes elliptiques

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Defining discrete log problem

Given a finite cyclic group $(G, +)$ of order N and two elements $g, h \in G$, find $x \in \mathbb{Z}$ such that

$$h = x \cdot g.$$

- Generic attacks - Pollard rho, Baby-step Giant-step, Kangaroo
- Index calculus attack : subexponential in $(\mathbb{Z}/p\mathbb{Z})*$.

Index calculus

1. Finding an appropriate *factor base* $\mathcal{B} = \{g_1, ..., g_k\}$, such that $\mathcal{B} \subseteq G$
2. Relation search phase : find relations of the form

$$[a_i]g \; + \; [b_i]h \; = \; \sum_{j=1}^{k} [c_{ij}]g_j$$

   for random integers a_i,b_i.
3. Linear algebra phase : having matrices $A = (a_i \;\; b_i)$ and $M = (c_{ij})$, find a kernel vector $v = (v_1 ... v_k)$ of the matrix M. Compute solution :

$$x = -(\sum_i a_i v_i)/(\sum_i b_i v_i)$$

Index calculus on binary elliptic curves

Let $\mathbb{F}_{2^n}$ be a finite field and E be an elliptic curve defined by

$$E : y^2 + xy = x^3 + ax^2 + b$$

with $a, b \in \mathbb{F}_{2^n}$ and n prime.

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$$E : y^2 + xy = x^3 + ax^2 + b$$

with $a, b \in \mathbb{F}_{2^n}$ and n prime.

1. Choice of an appropriate factor base $\mathcal{B}$
2. Point decomposition phase

   Find $P_1, \ldots, P_{m-1} \in \mathcal{B}$, such that, for $R \in E(\mathbb{F}_{2^n})$

   $$R = P_1 + \ldots + P_{m-1}$$

3. Linear algebra

Point Decomposition Problem (PDP)

Semaev's summation polynomials (2004)

$$S_2(X_1, X_2) = X_1 + X_2,$$
$$S_3(X_1, X_2, X_3) = X_1^2 X_2^2 + X_1^2 X_3^2 + X_1 X_2 X_3 + X_2^2 X_3^2 + b,$$

For $m \geq 4$

$$S_m(X_1, \ldots, X_m) =$$
$$Res_X(S_{m-k}(X_1, \ldots, X_{m-k-1}, X), S_{k+2}(X_{m-k}, \ldots, X_m, X))$$

Point Decomposition Problem (PDP)

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$$S_2(X_1, X_2) = X_1 + X_2,$$
$$S_3(X_1, X_2, X_3) = X_1^2X_2^2 + X_1^2X_3^2 + X_1X_2X_3 + X_2^2X_3^2 + b,$$

For $m \geq 4$

$$S_m(X_1, \ldots, X_m) =$$
$$Res_X(S_{m-k}(X_1, \ldots, X_{m-k-1}, X), S_{k+2}(X_{m-k}, \ldots, X_m, X))$$

Reducing the PDP to the problem of finding the roots of S_m

For $R, P_1, \ldots, P_{m-1} \in E(\mathbb{F}_{2^n})$

$$R + P_1 + \ldots + P_{m-1} = \mathcal{O} \iff S_m(\mathbf{x}_R, \mathbf{x}_{P_1}, \ldots, \mathbf{x}_{P_{m-1}}) = 0$$

Index calculus on elliptic curves over extension fields

Gaudry (2008), Diem (2009)

Choice of an appropriate factor base

When E is an elliptic curve defined over $\mathbb{F}_{q^n}$, with n small, the factor base is the set of points whose x-coordinate lies in $\mathbb{F}_q$.

Weil descent

Rewrite the equation $S_{n+1}(\mathbf{x}_R, X_1, \ldots, X_n) = 0$ as a system of n equations over $\mathbb{F}_q$.

Gaudry (2008)

Symmetrization

Rewrite S_m in terms of the elementary symmetric polynomials

$$\begin{aligned}
\mathbf{e}_1 &= \sum_{1 \le i_1 \le m} X_{i_1}, \\
\mathbf{e}_2 &= \sum_{1 \le i_1, i_2 \le m} X_{i_1} X_{i_2}, \\
&\dots \\
\mathbf{e}_m &= \prod_{1 \le i \le m} X_i.
\end{aligned}$$

PDP algebraic model

Yun-Ju *et al.* (2013)

Factor base for elliptic curves defined over $\mathbb{F}_{2^n}$, with n prime

An l-dimensional vector subspace V of $\mathbb{F}_{2^n}/\mathbb{F}_2$. When $l \sim \frac{n}{m}$ the system has a reasonable chance to have a solution.

Let t be a root of a defining polynomial of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$.

X_i-variables

$$X_1 = c_{1,0} + \ldots + c_{1,l-1}t^{l-1}$$
$$X_2 = c_{2,0} + \ldots + c_{2,l-1}t^{l-1}$$
$$\ldots$$
$$X_m = c_{m,0} + \ldots + c_{m,l-1}t^{l-1}$$

e_i-variables

$$e_1 = d_{1,0} + \ldots + d_{1,l-1}t^{l-1}$$
$$e_2 = d_{2,0} + \ldots + d_{2,2l-2}t^{2l-2}$$
$$\ldots$$
$$e_m = d_{m,0} + \ldots + d_{m,m(l-1)}t^{m(l-1)}$$

PDP algebraic model

Two sets of equations

- Equations defining symmetric polynomials

$$d_{1,0} = c_{1,0} + \ldots + c_{m,0}$$
$$d_{1,1} = c_{1,1} + \ldots + c_{m,1}$$
$$\ldots$$
$$d_{m,m(l-1)} = c_{1,l} \cdot \ldots \cdot c_{m,l}.$$

- Equations derived from the Weil descent

PDP algebraic model

Two sets of equations

- Equations defining symmetric polynomials

$$
\begin{aligned}
&d_{1,0} = c_{1,0} + \ldots + c_{m,0} \\
&d_{1,1} = c_{1,1} + \ldots + c_{m,1} \\
&\ldots \\
&d_{m,m(l-1)} = c_{1,l} \cdot \ldots \cdot c_{m,l}.
\end{aligned}
$$

- Equations derived from the Weil descent

The system is commonly solved using Gröbner basis methods.

Logical cryptanalysis

From the algebraic model to the SAT-reasoning model

Using SAT solvers as a cryptanalytic tool requires expressing the cryptographic problem as a Boolean formula in conjunctive normal form (CNF) - a conjunction ($\wedge$) of OR-clauses.

Example.

$$(\neg x_1 \vee x_2) \wedge$$
$$(\neg x_2 \vee x_4 \vee \neg x_5)) \wedge$$
$$(x_5 \vee x_6)$$

From the algebraic model to the SAT-reasoning model

XOR-enabled SAT solvers are adapted to read a formula in CNF-XOR form - a conjunction ($\wedge$) of OR-clauses and XOR-clauses.

Example.

$$(\neg x_1 \vee x_2) \wedge$$
$$(\neg x_2 \vee x_4 \vee \neg x_5)) \wedge$$
$$(x_1 \oplus x_5 \oplus x_6)$$

From the algebraic model to the SAT-reasoning model

Variables in $\mathbb{F}_2$:
$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6$.

Propositional variables:
$x_1, x_2, x_3, x_4, x_5, x_6$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$$\mathbf{x}_1 + \mathbf{x}_2 \cdot \mathbf{x}_4 + \mathbf{x}_5 \cdot \mathbf{x}_6 + 1 = 0$$
$$\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_4 + \mathbf{x}_5 + 1 = 0$$
$$\mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_2 \cdot \mathbf{x}_4 + 1 = 0$$
$$\mathbf{x}_2 + \mathbf{x}_5 + \mathbf{x}_2 \cdot \mathbf{x}_4 + \mathbf{x}_5 \cdot \mathbf{x}_6 + 1 = 0$$
$$\mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_6 + 1 = 0$$

$$(x_1 \oplus (x_2 \wedge x_4) \oplus (x_5 \wedge x_6)) \wedge$$
$$(x_1 \oplus x_2 \oplus x_4 \oplus x_5) \wedge$$
$$(x_3 \oplus x_4 \oplus (x_2 \wedge x_4)) \wedge$$
$$(x_2 \oplus x_5 \oplus (x_2 \wedge x_4) \oplus (x_5 \wedge x_6)) \wedge$$
$$(x_3 \oplus x_4 \oplus x_6)$$

From the algebraic model to the SAT-reasoning model

Add new variable x_7 to substitute the conjunction $x_2 \wedge x_4$. We have that

$$x_7 \Leftrightarrow (x_2 \wedge x_4)$$

$$\downarrow$$

$$(x_7 \Rightarrow (x_2 \wedge x_4)) \wedge ((x_2 \wedge x_4) \Rightarrow x_7)$$

$$\swarrow \qquad\qquad \searrow$$

$$\neg x_7 \vee (x_2 \wedge x_4) \qquad\qquad \neg(x_2 \wedge x_4) \vee x_7$$

$$\downarrow \qquad\qquad \downarrow$$

$$(\neg x_7 \vee x_2) \wedge (\neg x_7 \vee x_4) \qquad\qquad \neg x_2 \vee \neg x_4 \vee x_7$$

From the algebraic model to the SAT-reasoning model

Add new variable x_7 to substitute the conjunction $x_2 \wedge x_4$. We have that

$$x_7 \Leftrightarrow (x_2 \wedge x_4)$$

$$(x_7 \Rightarrow (x_2 \wedge x_4)) \wedge ((x_2 \wedge x_4) \Rightarrow x_7)$$

$$\neg x_7 \vee (x_2 \wedge x_4) \qquad \neg(x_2 \wedge x_4) \vee x_7$$

$$(\neg x_7 \vee x_2) \wedge (\neg x_7 \vee x_4) \qquad \neg x_2 \vee \neg x_4 \vee x_7$$

From the algebraic model to the SAT-reasoning model

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$$
\begin{aligned}
&(x_1 \oplus (x_2 \wedge x_4) \oplus (x_5 \wedge x_6)) \wedge \\
&(x_1 \oplus x_2 \oplus x_4 \oplus x_5) \wedge \\
&(x_3 \oplus x_4 \oplus (x_2 \wedge x_4)) \wedge \\
&(x_2 \oplus x_5 \oplus (x_2 \wedge x_4) \oplus (x_5 \wedge x_6)) \wedge \\
&(x_3 \oplus x_4 \oplus x_6)
\end{aligned}
$$

$$
\begin{aligned}
&(\neg x_7 \vee x_2) \wedge \\
&(\neg x_7 \vee x_4) \wedge \\
&(\neg x_2 \vee \neg x_4 \vee x_7) \wedge \\
&(\neg x_8 \vee x_5) \wedge \\
&(\neg x_8 \vee x_6) \wedge \\
&(\neg x_5 \vee \neg x_6 \vee x_8) \wedge \\
&(x_1 \oplus x_7 \oplus x_8) \wedge \\
&(x_1 \oplus x_2 \oplus x_4 \oplus x_5) \wedge \\
&(x_3 \oplus x_4 \oplus x_7) \wedge \\
&(x_2 \oplus x_5 \oplus x_7 \oplus x_8) \wedge \\
&(x_3 \oplus x_4 \oplus x_6)
\end{aligned}
$$

Unit propagation

Assigning literal l to TRUE will lead to :

1. Every clause containing l is removed (since the clause is satisfied).
2. In every clause that contains $\neg l$ this literal is deleted (since it can not contribute to the clause being satisfied).

- Propagation - obtaining a *unit clause* (clause containing a single literal) $\rightarrow$ the remaining literal is set to TRUE.
- Conflict - it exists at least one clause with all literals assigned to FALSE.

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$$
\begin{aligned}
&(\neg x_7 \vee x_2) \wedge \\
&(\neg x_7 \vee x_4) \wedge \\
&(\neg x_2 \vee \neg x_4 \vee x_7) \wedge \\
&(\neg x_8 \vee x_5) \wedge \\
&(\neg x_8 \vee x_6) \wedge \\
&(\neg x_5 \vee \neg x_6 \vee x_8) \wedge \\
&(x_1 \oplus x_7 \oplus x_8) \wedge \\
&(x_1 \oplus x_2 \oplus x_4 \oplus x_5) \wedge \\
&(x_3 \oplus x_4 \oplus x_7) \wedge \\
&(x_2 \oplus x_5 \oplus x_7 \oplus x_8) \wedge \\
&(x_3 \oplus x_4 \oplus x_6)
\end{aligned}
$$

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$$(\neg x_7 \vee x_2) \wedge$$
$$(\neg x_7 \vee x_4) \wedge$$
$$(\neg x_2 \vee \neg x_4 \vee x_7) \wedge$$
$$(\neg x_8 \vee x_5) \wedge$$
$$(\neg x_8 \vee x_6) \wedge$$
$$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$$
$$(x_1 \oplus x_7 \oplus x_8) \wedge$$
$$(x_1 \oplus x_2 \oplus x_4 \oplus x_5) \wedge$$
$$(x_3 \oplus x_4 \oplus x_7) \wedge$$
$$(x_2 \oplus x_5 \oplus x_7 \oplus x_8) \wedge$$
$$(x_3 \oplus x_4 \oplus x_6)$$

Assigned: $x_1 \leftarrow F$;

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_7 \vee x_2) \wedge$

$(\neg x_7 \vee x_4) \wedge$

$(\neg x_2 \vee \neg x_4 \vee x_7) \wedge$

$(\neg x_8 \vee x_5) \wedge$

$(\neg x_8 \vee x_6) \wedge$

$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$

$(\cancel{x_1} \oplus x_7 \oplus x_8) \wedge$

$(\cancel{x_1} \oplus x_2 \oplus x_4 \oplus x_5) \wedge$

$(x_3 \oplus x_4 \oplus x_7) \wedge$

$(x_2 \oplus x_5 \oplus x_7 \oplus x_8) \wedge$

$(x_3 \oplus x_4 \oplus x_6)$

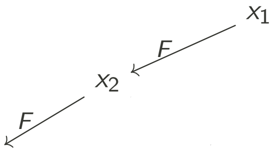
Assigned: $x_1 \leftarrow F$;

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_7 \vee x_2) \wedge$
$(\neg x_7 \vee x_4) \wedge$
$(\neg x_2 \vee \neg x_4 \vee x_7) \wedge$
$(\neg x_8 \vee x_5) \wedge$
$(\neg x_8 \vee x_6) \wedge$
$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$
$(x_7 \oplus x_8) \wedge$
$(x_2 \oplus x_4 \oplus x_5) \wedge$
$(x_3 \oplus x_4 \oplus x_7) \wedge$
$(x_2 \oplus x_5 \oplus x_7 \oplus x_8) \wedge$
$(x_3 \oplus x_4 \oplus x_6)$



Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow F$;

Propagation: $x_7 \leftarrow F$;

Solving process example

Propositional variables:

x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_7 \vee x_2) \wedge$

$(\neg x_7 \vee x_4) \wedge$

$(\neg x_2 \vee \neg x_4 \vee x_7) \wedge$

$(\neg x_8 \vee x_5) \wedge$

$(\neg x_8 \vee x_6) \wedge$

$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$

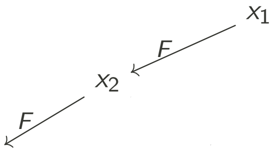
$(x_7 \oplus x_8) \wedge$

$(x_2 \oplus x_4 \oplus x_5) \wedge$

$(x_3 \oplus x_4 \oplus x_7) \wedge$

$(x_2 \oplus x_5 \oplus x_7 \oplus x_8) \wedge$

$(x_3 \oplus x_4 \oplus x_6)$



Assigned: $x_1 \leftarrow F; x_2 \leftarrow F; x_7 \leftarrow F;$

Propagation: $x_8 \leftarrow T;$

Solving process example

Propositional variables:

x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_8 \vee x_5) \wedge$

$(\neg x_8 \vee x_6) \wedge$

$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$

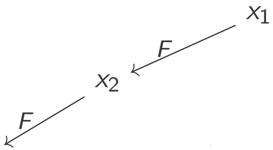
$(x_8) \wedge$

$(x_4 \oplus x_5) \wedge$

$(x_3 \oplus x_4) \wedge$

$(x_5 \oplus x_8) \wedge$

$(x_3 \oplus x_4 \oplus x_6)$



Assigned: $x_1 \leftarrow F; x_2 \leftarrow F; x_7 \leftarrow F;$ $x_8 \leftarrow T;$

Propagation: $x_5 \leftarrow T; x_6 \leftarrow T;$

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

~~$(\neg x_8 \vee x_5) \wedge$~~

~~$(\neg x_8 \vee x_6) \wedge$~~

~~$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$~~

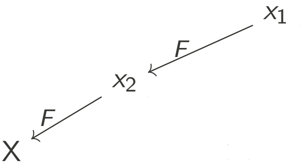
$(\cancel{x_8}\, T) \wedge$

$(x_4 \oplus \cancel{x_5}\, T) \wedge$

$(x_3 \oplus x_4) \wedge$

$(\cancel{x_5}\, T \oplus \cancel{x_8}\, T) \wedge$

$(x_3 \oplus x_4 \oplus \cancel{x_6}\, T)$



Assigned: $x_1 \leftarrow F; x_2 \leftarrow F; x_7 \leftarrow F;$
$x_8 \leftarrow T; x_5 \leftarrow T; x_6 \leftarrow T;$

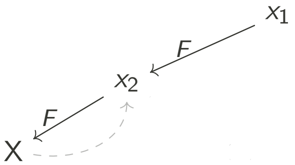
Conflict on fourth XOR-clause.

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$$
\begin{aligned}
&(\neg x_7 \vee x_2) \wedge \\
&(\neg x_7 \vee x_4) \wedge \\
&(\neg x_2 \vee \neg x_4 \vee x_7) \wedge \\
&(\neg x_8 \vee x_5) \wedge \\
&(\neg x_8 \vee x_6) \wedge \\
&(\neg x_5 \vee \neg x_6 \vee x_8) \wedge \\
&(x_7 \oplus x_8) \wedge \\
&(x_2 \oplus x_4 \oplus x_5) \wedge \\
&(x_3 \oplus x_4 \oplus x_7) \wedge \\
&(x_2 \oplus x_5 \oplus x_7 \oplus x_8) \wedge \\
&(x_3 \oplus x_4 \oplus x_6)
\end{aligned}
$$



Assigned: $x_1 \leftarrow F$;

Backtrack.

Solving process example

Propositional variables:

x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 with truth values in $\{\text{TRUE}, \text{FALSE}\}$

~~$(\neg x_7 \vee x_2)$~~ $\wedge$

$(\neg x_7 \vee x_4) \wedge$

$($~~$\neg x_2$~~$\vee \neg x_4 \vee x_7) \wedge$

$(\neg x_8 \vee x_5) \wedge$

$(\neg x_8 \vee x_6) \wedge$

$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$

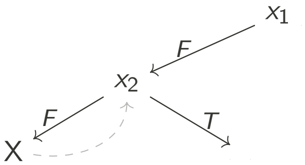
$(x_7 \oplus x_8) \wedge$

$($~~x_2~~$T \oplus x_4 \oplus x_5) \wedge$

$(x_3 \oplus x_4 \oplus x_7) \wedge$

$($~~x_2~~$T \oplus x_5 \oplus x_7 \oplus x_8) \wedge$

$(x_3 \oplus x_4 \oplus x_6)$

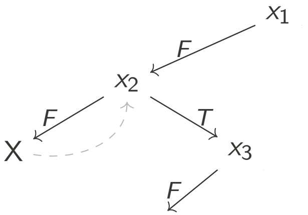


Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$;

Propositional variables:

x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$$(\neg x_7 \vee x_4) \wedge$$
$$(\neg x_4 \vee x_7) \wedge$$
$$(\neg x_8 \vee x_5) \wedge$$
$$(\neg x_8 \vee x_6) \wedge$$
$$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$$
$$(x_7 \oplus x_8) \wedge$$
$$(T \oplus x_4 \oplus x_5) \wedge$$
$$(\cancel{x_3} \oplus x_4 \oplus x_7) \wedge$$
$$(T \oplus x_5 \oplus x_7 \oplus x_8) \wedge$$
$$(\cancel{x_3} \oplus x_4 \oplus x_6)$$



Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$; $x_3 \leftarrow F$;

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_7 \vee \cancel{x_4}) \wedge$

$\cancel{(\neg x_4 \vee x_7)} \wedge$

$(\neg x_8 \vee x_5) \wedge$

$(\neg x_8 \vee x_6) \wedge$

$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$

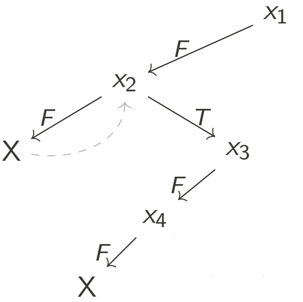
$(x_7 \oplus x_8) \wedge$

$(T \oplus \cancel{x_4} \oplus x_5) \wedge$

$(\cancel{x_4} \oplus x_7) \wedge$

$(T \oplus x_5 \oplus x_7 \oplus x_8) \wedge$

$(\cancel{x_4} \oplus x_6)$



Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$; $x_3 \leftarrow F$; $x_4 \leftarrow F$;

Propagation: $x_7 \leftarrow F$; $x_7 \leftarrow T$; Conflict.

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

~~$(\neg x_7 \vee x_4)$~~ $\wedge$

$($~~$\neg x_4$~~ $\vee x_7) \wedge$

$(\neg x_8 \vee x_5) \wedge$

$(\neg x_8 \vee x_6) \wedge$

$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$

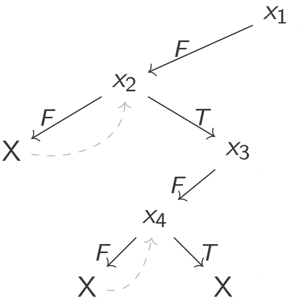
$(x_7 \oplus x_8) \wedge$

$(T \oplus$ ~~x_4~~ $T \oplus x_5) \wedge$

$($~~x_4~~ $T \oplus x_7) \wedge$

$(T \oplus x_5 \oplus x_7 \oplus x_8) \wedge$

$($~~x_4~~ $T \oplus x_6)$



Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$; $x_3 \leftarrow F$; $x_4 \leftarrow F$;

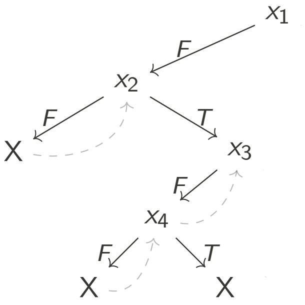
Propagation: $x_7 \leftarrow T$; $x_7 \leftarrow F$; ...
Conflict.

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$$
\begin{aligned}
&(\neg x_7 \vee x_4) \wedge \\
&(\neg x_4 \vee x_7) \wedge \\
&(\neg x_8 \vee x_5) \wedge \\
&(\neg x_8 \vee x_6) \wedge \\
&(\neg x_5 \vee \neg x_6 \vee x_8) \wedge \\
&(x_7 \oplus x_8) \wedge \\
&(T \oplus x_4 \oplus x_5) \wedge \\
&(x_3 \oplus x_4 \oplus x_7) \wedge \\
&(T \oplus x_5 \oplus x_7 \oplus x_8) \wedge \\
&(x_3 \oplus x_4 \oplus x_6)
\end{aligned}
$$



Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$;

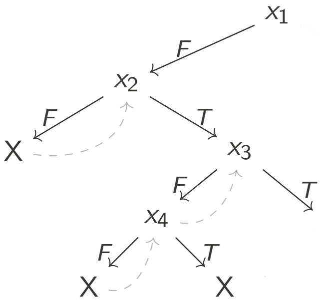
Backtrack

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_7 \lor x_4) \land$
$(\neg x_4 \lor x_7) \land$
$(\neg x_8 \lor x_5) \land$
$(\neg x_8 \lor x_6) \land$
$(\neg x_5 \lor \neg x_6 \lor x_8) \land$
$(x_7 \oplus x_8) \land$
$(T \oplus x_4 \oplus x_5) \land$
$(\cancel{x_3}\, T \oplus x_4 \oplus x_7) \land$
$(T \oplus x_5 \oplus x_7 \oplus x_8) \land$
$(\cancel{x_3}\, T \oplus x_4 \oplus x_6)$



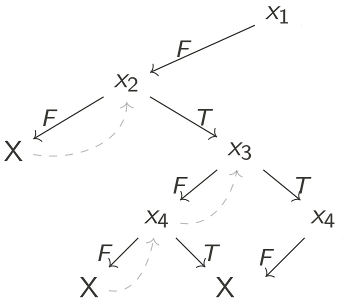
Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$; $x_3 \leftarrow T$;

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_7 \vee x_4) \wedge$
$(\neg x_4 \vee x_7) \wedge$
$(\neg x_8 \vee x_5) \wedge$
$(\neg x_8 \vee x_6) \wedge$
$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$
$(x_7 \oplus x_8) \wedge$
$(T \oplus x_4 \oplus x_5) \wedge$
$(T \oplus x_4 \oplus x_7) \wedge$
$(T \oplus x_5 \oplus x_7 \oplus x_8) \wedge$
$(T \oplus x_4 \oplus x_6)$



Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$; $x_3 \leftarrow T$;

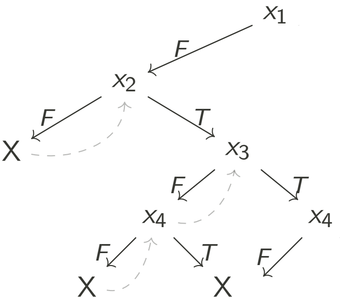
Propagated: $x_7 \leftarrow F$;

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$$(\neg x_8 \lor x_5) \land$$
$$(\neg x_8 \lor x_6) \land$$
$$(\neg x_5 \lor \neg x_6 \lor x_8) \land$$
$$(\cancel{x_7} \oplus x_8) \land$$
$$(T \oplus \cancel{x_4} \oplus x_5) \land$$
$$(T \oplus \cancel{x_4} \oplus \cancel{x_7}) \land$$
$$(T \oplus x_5 \oplus \cancel{x_7} \oplus x_8) \land$$
$$(T \oplus \cancel{x_4} \oplus x_6)$$



Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$; $x_3 \leftarrow T$; $x_4 \leftarrow F$; $x_7 \leftarrow F$;

Propagated: $x_8 \leftarrow T$; $x_5 \leftarrow F$; $x_6 \leftarrow F$;

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_8 \vee x_5) \wedge$

$(\neg x_8 \vee x_6) \wedge$

$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$

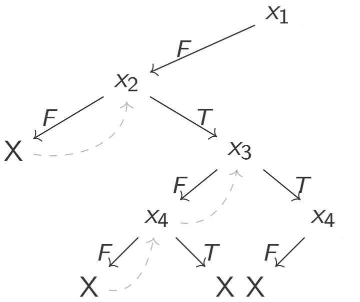
$(x_8 T) \wedge$

$(T \oplus x_5) \wedge$

$(T) \wedge$

$(T \oplus x_5 \oplus x_8 T) \wedge$

$(T \oplus x_6)$



Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$; $x_3 \leftarrow T$; $x_4 \leftarrow F$; $x_7 \leftarrow F$; $x_8 \leftarrow T$; $x_5 \leftarrow F$; $x_6 \leftarrow F$;

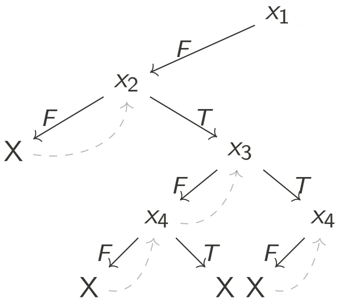
Conflict.

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE, FALSE}\}$

$(\neg x_7 \vee x_4) \wedge$
$(\neg x_4 \vee x_7) \wedge$
$(\neg x_8 \vee x_5) \wedge$
$(\neg x_8 \vee x_6) \wedge$
$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$
$(x_7 \oplus x_8) \wedge$
$(T \oplus x_4 \oplus x_5) \wedge$
$(T \oplus x_4 \oplus x_7) \wedge$
$(T \oplus x_5 \oplus x_7 \oplus x_8) \wedge$
$(T \oplus x_4 \oplus x_6)$



Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$; $x_3 \leftarrow T$;

Backtrack.

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$\cancel{(\neg x_7 \lor x_4)} \land$

$(\cancel{\neg x_4} \lor x_7) \land$

$(\neg x_8 \lor x_5) \land$

$(\neg x_8 \lor x_6) \land$

$(\neg x_5 \lor \neg x_6 \lor x_8) \land$

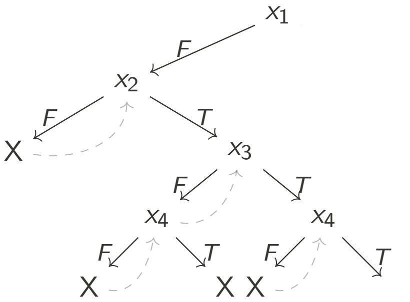
$(\cancel{x_7}\, T \oplus x_8) \land$

$(T \oplus \cancel{x_4}\, T \oplus x_5) \land$

$(T \oplus \cancel{x_4}\, T \oplus \cancel{x_7}\, T) \land$

$(T \oplus x_5 \oplus \cancel{x_7}\, T \oplus x_8) \land$

$(T \oplus \cancel{x_4}\, T \oplus x_6)$



Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$; $x_3 \leftarrow T$; $x_4 \leftarrow T$;

Propagation: $x_7 \leftarrow T$; $x_5 \leftarrow T$; $x_6 \leftarrow T$; $x_8 \leftarrow F$;

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_8 \vee x_5) \wedge$

$(\neg x_8 \vee x_6) \wedge$

$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$

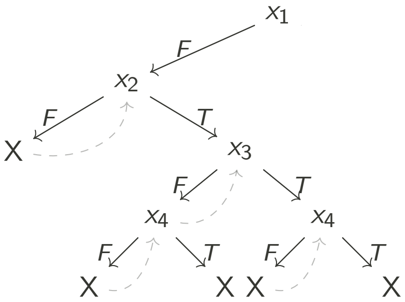
$(T) \wedge$

$(T) \wedge$

$(T) \wedge$

$(T) \wedge$

$(T) \wedge$



Assigned: $x_1 \leftarrow F$; $x_2 \leftarrow T$; $x_3 \leftarrow T$; $x_4 \leftarrow T$; $x_7 \leftarrow T$; $x_5 \leftarrow T$; $x_6 \leftarrow T$; $x_8 \leftarrow F$;

Conflict.

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_7 \vee x_2) \wedge$

$(\neg x_7 \vee x_4) \wedge$

$(\neg x_2 \vee \neg x_4 \vee x_7) \wedge$

$(\neg x_8 \vee x_5) \wedge$

$(\neg x_8 \vee x_6) \wedge$

$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$

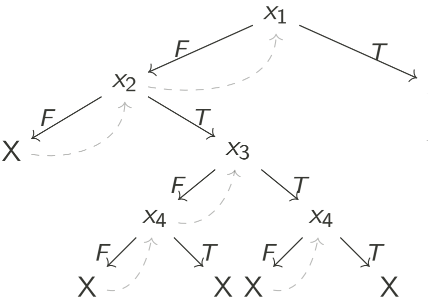
$(\cancel{x_1}\, T \oplus x_7 \oplus x_8) \wedge$

$(\cancel{x_1}\, T \oplus x_2 \oplus x_4 \oplus x_5) \wedge$

$(x_3 \oplus x_4 \oplus x_7) \wedge$

$(x_2 \oplus x_5 \oplus x_7 \oplus x_8) \wedge$

$(x_3 \oplus x_4 \oplus x_6)$



Assigned: $x_1 \leftarrow T$;

Solving process example

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_7 \lor x_2) \land$

$(\neg x_7 \lor x_4) \land$

$(\neg x_2 \lor \neg x_4 \lor x_7) \land$

$(\neg x_8 \lor x_5) \land$

$(\neg x_8 \lor x_6) \land$

$(\neg x_5 \lor \neg x_6 \lor x_8) \land$

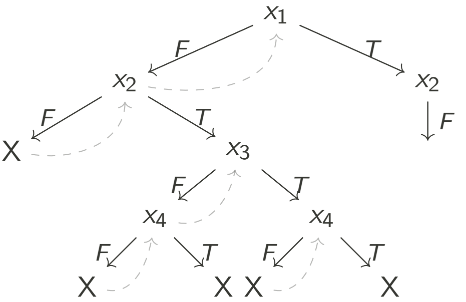
$(T \oplus x_7 \oplus x_8) \land$

$(T \oplus x_2 \oplus x_4 \oplus x_5) \land$

$(x_3 \oplus x_4 \oplus x_7) \land$

$(x_2 \oplus x_5 \oplus x_7 \oplus x_8) \land$

$(x_3 \oplus x_4 \oplus x_6)$



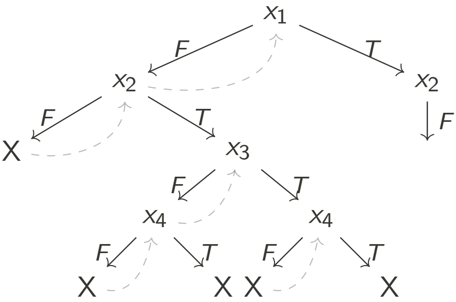
Assigned: $x_1 \leftarrow T$; $x_2 \leftarrow F$;

Propagation: $x_7 \leftarrow F$;

Solving process example

Propositional variables:
$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

$(\neg x_8 \vee x_5) \wedge$
$(\neg x_8 \vee x_6) \wedge$
$(\neg x_5 \vee \neg x_6 \vee x_8) \wedge$
$(T \oplus \cancel{x_7} \oplus x_8) \wedge$
$(T \oplus \cancel{x_2} \oplus x_4 \oplus x_5) \wedge$
$(x_3 \oplus x_4 \oplus \cancel{x_7}) \wedge$
$(\cancel{x_2} \oplus x_5 \oplus \cancel{x_7} \oplus x_8) \wedge$
$(x_3 \oplus x_4 \oplus x_6)$



Assigned: $x_1 \leftarrow T$; $x_2 \leftarrow F$; $x_7 \leftarrow F$;

Propagation: $x_8 \leftarrow F$;

Propositional variables:

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$ with truth values in $\{\text{TRUE}, \text{FALSE}\}$

~~$(\neg x_8 \vee x_5) \wedge$~~

~~$(\neg x_8 \vee x_6) \wedge$~~

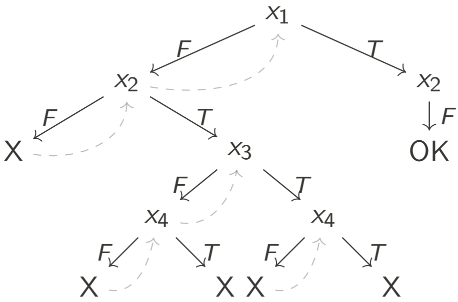
$(\neg x_5 \vee \neg x_6 \vee \cancel{x_8}) \wedge$

$(T \oplus x_4 \oplus x_5) \wedge$

$(x_3 \oplus x_4) \wedge$

$(x_5 \oplus \cancel{x_8}) \wedge$

$(x_3 \oplus x_4 \oplus x_6)$



Assigned: $x_1 \leftarrow T$; $x_2 \leftarrow F$; $x_7 \leftarrow F$; $x_8 \leftarrow F$;

Propagation: $x_5 \leftarrow T$; ... $x_4 \leftarrow T$; ... $x_3 \leftarrow F$; ... $x_6 \leftarrow F$;

DPLL algorithm

Davis-Putnam-Logemann-Loveland (DPLL) algorithm

Building a binary search-tree of height equivalent (at worst) to the number of variables.

WDSat solver

- DPLL-based.
- Composed of three reasoning modules: CNF module, XORSET module and XORGAUSS module.

Branching variables

1. Our DPLL-based algorithm only makes assignments on variables that are present in the initial Boolean polynomial system. Substitution variables are propagated as a consequence.

Branching variables

1. Our DPLL-based algorithm only makes assignments on variables that are present in the initial Boolean polynomial system. Substitution variables are propagated as a consequence.
2. We only reason on X_i-variables. e_i-variables are propagated.

X_i-variables

$$X_1 = c_{1,0} + \ldots + c_{1,l-1} t^{l-1}$$
$$X_2 = c_{2,0} + \ldots + c_{2,l-1} t^{l-1}$$
$$\ldots$$
$$X_m = c_{m,0} + \ldots + c_{m,l-1} t^{l-1}$$

e_i-variables

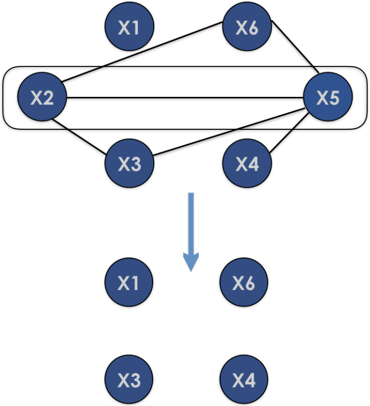
$$e_1 = d_{1,0} + \ldots + d_{1,l-1} t^{l-1}$$
$$e_2 = d_{2,0} + \ldots + d_{2,2l-2} t^{2l-2}$$
$$\ldots$$
$$e_m = d_{m,0} + \ldots + d_{m,m(l-1)} t^{m(l-1)}$$

MVC preprocessing technique

$$\mathbf{x}_1 + \mathbf{x}_2\mathbf{x}_3 + \mathbf{x}_4 + \mathbf{x}_4\mathbf{x}_5 = 0$$
$$\mathbf{x}_1 + \mathbf{x}_2\mathbf{x}_3 = 0$$
$$\mathbf{x}_1 + \mathbf{x}_3\mathbf{x}_5 + \mathbf{x}_6 = 0$$
$$\mathbf{x}_1 + \mathbf{x}_2\mathbf{x}_5\mathbf{x}_6 + \mathbf{x}_6 = 0$$

↓

$$\mathbf{x}_1 + \mathbf{x}_3 = 0$$
$$\mathbf{x}_1 + \mathbf{x}_3 + \mathbf{x}_6 = 0$$
$$\mathbf{x}_1 = 0.$$



MVC of third summation polynomial

$$S_3(X_1, X_2, X_3) = X_1^2 X_2^2 + X_1^2 X_3^2 + X_1 X_2 X_3 + X_2^2 X_3^2 + 1,$$

$$X_1 = c_{1,0} + \ldots + c_{1,l-1} t^{l-1}$$
$$X_2 = c_{2,0} + \ldots + c_{2,l-1} t^{l-1}$$
X_3 is a constant

MVC of third summation polynomial

$$S_3(X_1, X_2, X_3) = X_1^2 X_2^2 + X_1^2 X_3^2 + X_1 X_2 X_3 + X_2^2 X_3^2 + 1,$$

	1	2	3	4	5	6	7	8	9	0
1	○	○	○	○	○	●	●	●	●	●
2	○	○	○	○	○	●	●	●	●	●
3	○	○	○	○	○	●	●	●	●	●
4	○	○	○	○	○	●	●	●	●	●
5	○	○	○	○	○	●	●	●	●	●
6	●	●	●	●	●	○	○	○	○	○
7	●	●	●	●	●	○	○	○	○	○
8	●	●	●	●	●	○	○	○	○	○
9	●	●	●	●	●	○	○	○	○	○
0	●	●	●	●	●	○	○	○	○	○

Figure: Monomials connectivity graph derived from the model of S_3 when $l = 5$

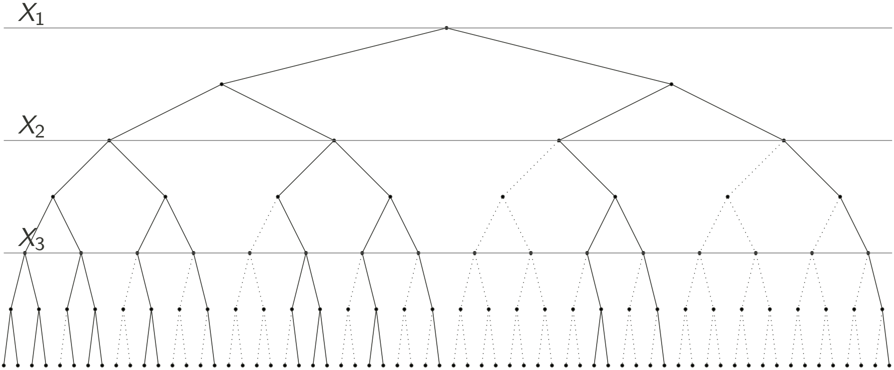
MVC of third summation polynomial

$$S_3(X_1, X_2, X_3) = X_1^2 X_2^2 + X_1^2 X_3^2 + X_1 X_2 X_3 + X_2^2 X_3^2 + 1,$$

- MVC is $\{c_{1,0}, \ldots, c_{1,l-1}\}$ or $\{c_{2,0}, \ldots, c_{2,l-1}\}$.
- Using the MVC preprocessing technique, the complexity of a point decomposition drops from $O(2^{2l})$ to $O(2^l)$.
- Unfortunately, we do not observe this with higher degree summation polynomials.

- Exploit the symmetry of Semaev's summation polynomials: when $X_1, ..., X_m$ is a solution, all permutations of this set are a solution as well.
- Establish the following constraint $X_1 \leq X_2 \leq \ldots \leq X_m$.
- Implement constraint in the solver using a tree-pruning-like technique.
- Optimize the complexity by a factor of $m!$.

WDSat - breaking symmetry



Experimental results

Third summation polynomial
$n = 41,\ l = 20$

Solving approach	SAT		UNSAT	
	Runtime (s)	#Conflicts	Runtime (s)	#Conflicts
Gröbner	16.8	*N/A*	18.7	*N/A*
MiniSat	> 600		> 600	
Glucose	> 600		> 600	
MapleLCMDistChronoBT	> 600		> 600	
CaDiCaL	> 600		> 600	
CryptoMiniSat	29.0	226668	84.3	627539
WDSat+XG-EXT+MVC	**4.2**	**27684**	**13.5**	**86152**

Table: Comparing Gröbner basis and SAT-based approaches for solving the PDP. Running times are in seconds.

Experimental results

Fourth summation polynomial

			SATisfiable			UNSATisfiable		
Approach	l	n	Runtime	#Conflicts	Memory	Runtime	#Conflicts	Memory
Gröbner basis	6	17	207.220	NA	3601	142.119	NA	3291
		19	215.187	NA	3940	155.765	NA	4091
	7	19	3854.708	NA	38763	2650.696	NA	38408
		23	3128.844	NA	35203	2286.136	NA	35162
WDSat	6	17	.601	49117	1.4	3.851	254686	1.4
		19	.470	38137	1.4	3.913	255491	1.4
	7	19	9.643	534867	16.7	44.107	2073089	16.7
		23	9.303	477632	16.7	47.347	2067168	16.7
WDSat breaking-sym	6	17	.220	17792	1.4	.605	43875	1.4
		19	.243	19166	1.4	.639	44034	1.4
	7	19	2.205	130062	1.4	6.859	351353	1.4
		23	3.555	189940	1.4	7.478	350257	1.4

Table: Comparing the WDSat approach with the Gröbner basis approach for solving the PDP. Running times are in seconds and memory is in MB.

Conclusion

- When solving the PDP for prime degree extension fields of $\mathbb{F}_2$, Gröbner basis methods should be replaced with a SAT-based approach.
- Our CNF-XOR model with the dedicated SAT-solver, WDSAT, yields significantly faster running times than all other algebraic and SAT-based approaches.
- Extending the WDSAT solver with our symmetry breaking technique optimizes the resolution of the PDP by a factor of m!.

Papers and open source code

1. **Parity (XOR) Reasoning for the Index Calculus Attack** (CP 2020)

   https://arxiv.org/abs/2001.11229

   https://github.com/mtrimoska/WDSat

2. **A SAT-Based Approach for Index Calculus on Binary Elliptic Curves** (AfricaCrypt 2020)

   https://eprint.iacr.org/2019/313

   https://github.com/mtrimoska/EC-Index-Calculus-Benchmarks