Hidden Structures in Quantum Cryptanalysis

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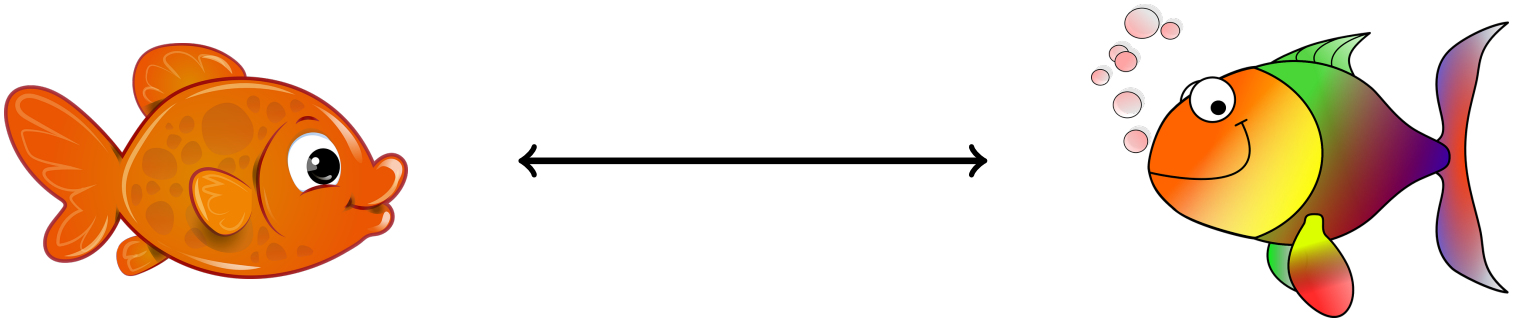
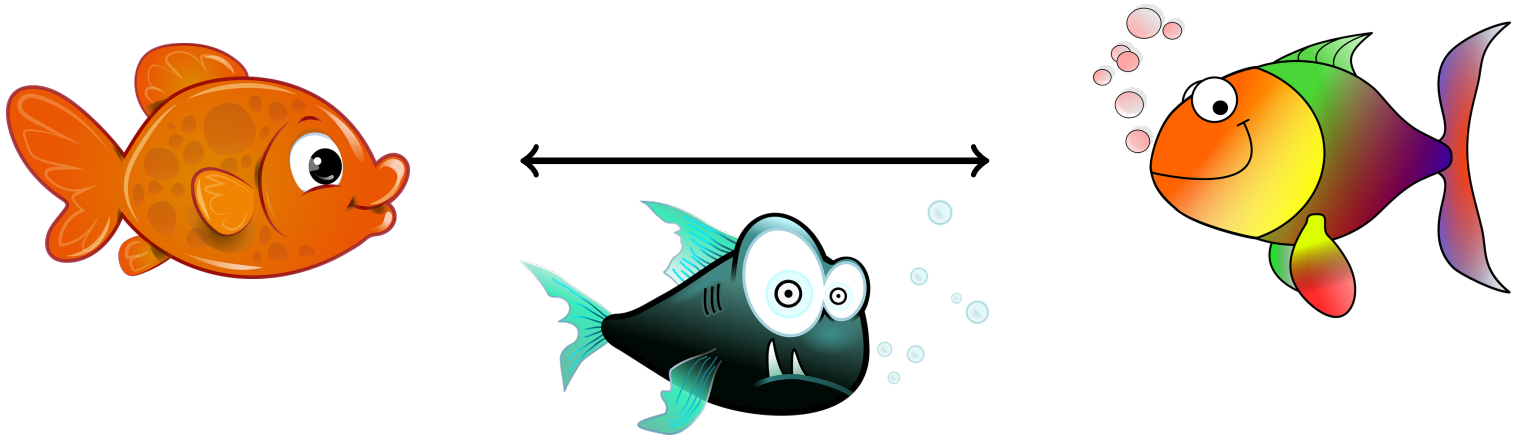
Cryptography

Aims at enabling secure communications



Cryptography

Aims at enabling secure communications

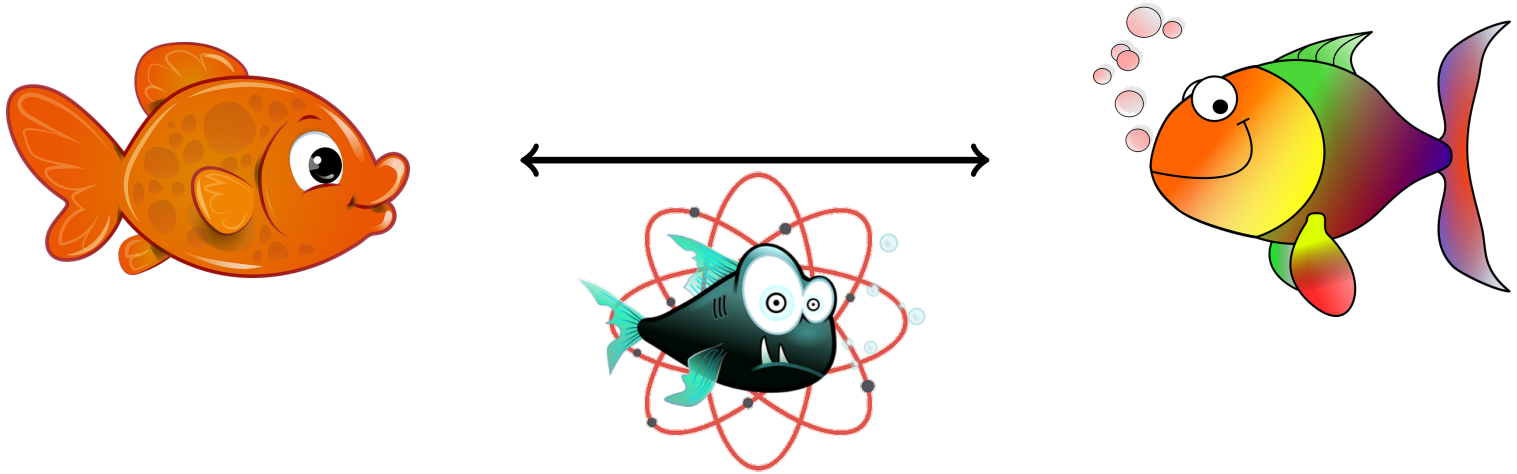


An adversary appears!

She wants to attack the communication.

Cryptography

Aims at enabling secure communications



A quantum adversary appears!

She wants to attack the communication.

Quantum computing

Classical computing

x	Input
$\downarrow$	
$a, b, \ldots$	Intermediate values
$\downarrow$	
y	Final result

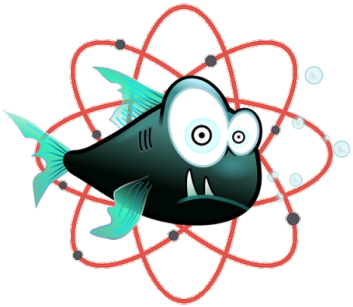
Quantum computing

$\lvert x\rangle$	Input
$\downarrow$	
$\lvert a, b, \ldots\rangle$	Intermediate state
$\downarrow$	
$\lvert y\rangle \longrightarrow y$	Final measurement

Differences

- More possibilities $|0\rangle$, $|1\rangle$, $|0\rangle - |1\rangle \ldots$
- Reversible computing
- New operators $H : |b\rangle \mapsto \frac{1}{\sqrt{2}} \left(|0\rangle + (-1)^b |1\rangle\right)$

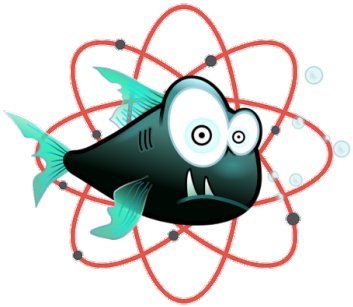
Quantum Adversary



First model: Classical queries (Q1)

- The adversary can perform quantum computing
- The adversary is restricted to classical queries : $x \mapsto f(x)$

Quantum Adversary



First model: Classical queries (Q1)

- The adversary can perform quantum computing
- The adversary is restricted to classical queries : $x \mapsto f(x)$

Second model: Quantum queries (Q2)

- The adversary can perform quantum computing
- Can do quantum queries : $\sum_x |x\rangle |0\rangle \mapsto \sum_x |x\rangle |f(x)\rangle$

Reference quantum algorithms

Shor's algorithm [Sho94]

Solves the factorization and discrete logarithm problems in polynomial time.

Quantum Search [Gro96]

- Search space of size $2^{\mathbf{n}}$, quantum test circuit T_χ for χ
- Find $\mathbf{s}$ such that $\chi(\mathbf{s}) = 1$ in $2^{\mathbf{n}/2}$ iterations of T_χ

Quantum Collision search [BHT98]

- $f : \{0,1\}^{\mathbf{n}} \to \{0,1\}^{\mathbf{n}}$, find $x \neq y : f(x) = f(y)$
- Classically: $2^{\mathbf{n}/2}$ operations
- Brassard-Høyer-Tapp: $2^{\mathbf{n}/3}$ operations and $2^{\mathbf{n}/3}$ quantum memory

Outline

Simon's algorithm

Simon's problem

Simon's problem

- $f : \{0,1\}^{\mathbf{n}} \to \{0,1\}^{\mathbf{n}}$
- $s \in \{0,1\}^{\mathbf{n}}$
- $\forall x, y, f(y) = f(x) \Leftrightarrow x \oplus y \in \{0, s\}$
- f hides the period s
- Goal : find s, given oracle access to f.

Classical resolution

Find a collision, in $\Omega\left(2^{\mathbf{n}/2}\right)$ samples.

Quantum resolution

Simon's algorithm, in $\mathcal{O}(\mathbf{n})$ quantum queries, $\mathcal{O}\left(\mathbf{n}^3\right)$ classical operations

Simon's algorithm [Sim94]

Quantum circuit

- *Start from* $|0\rangle\,|0\rangle$

Simon's algorithm [Sim94]

Quantum circuit

- Start from $|0\rangle |0\rangle$
- *Apply H:* $\frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle |0\rangle$

Simon's algorithm [Sim94]

Quantum circuit

- Start from $|0\rangle |0\rangle$
- Apply H: $\frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle |0\rangle$
- *Apply* O_f*:* $\frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle |f(x)\rangle$

Simon's algorithm [Sim94]

Quantum circuit

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- *Measure the second register: get* $f(x_0)$ *and project to* $\frac{1}{\sqrt{2}}(|x_0\rangle + |x_0 \oplus s\rangle)$

Simon's algorithm [Sim94]

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- Measure the second register: get $f(x_0)$ and project to $\frac{1}{\sqrt{2}}(|x_0\rangle + |x_0 \oplus s\rangle)$
- *Reapply H:* $\frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} (-1)^{x_0 \cdot y} |y\rangle + (-1)^{(x_0 \oplus s) \cdot y} |y\rangle$

Simon's algorithm [Sim94]

Quantum circuit

- Start from $|0\rangle|0\rangle$
- Apply H: $\frac{1}{2^{n/2}}\sum_{x=0}^{2^n-1}|x\rangle|0\rangle$
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- Measure the second register: get $f(x_0)$ and project to $\frac{1}{\sqrt{2}}(|x_0\rangle + |x_0 \oplus s\rangle)$
- Reapply H: $\frac{1}{2^{n/2}}\sum_{y=0}^{2^n-1}(-1)^{x_0\cdot y}|y\rangle + (-1)^{(x_0\oplus s)\cdot y}|y\rangle$
- *The state is* $\frac{1}{2^{n/2}}\sum_{y=0}^{2^n-1}(-1)^{x_0\cdot y}(1+(-1)^{s\cdot y})|y\rangle$

The y_0 we measure must satisfy $1+(-1)^{s\cdot y_0} \neq 0 \Rightarrow y_0 \cdot s = 0$.

Simon's algorithm [Sim94]

Simon's problem

- $f : \{0,1\}^{\mathbf{n}} \to \{0,1\}^{\mathbf{n}}$, $\mathbf{s} \in \{0,1\}^{\mathbf{n}}$
- $\forall x, y, f(y) = f(x) \Leftrightarrow x \oplus y \in \{0, \mathbf{s}\}$
- Goal : find s, given oracle access to f.

Simon's algorithm

- Superposition queries $\sum_x |x\rangle |f(x)\rangle$
- Sample y: $\mathbf{s} \cdot y = 0$
- Repeat $O(\mathbf{n})$ times and solve the system

Generalizations

- Works if $\forall x, y, f(y) = f(x) \Leftarrow x \oplus y \in \{0, s\}$
- Works with multiple periods

Simon-based cryptanalysis

General idea

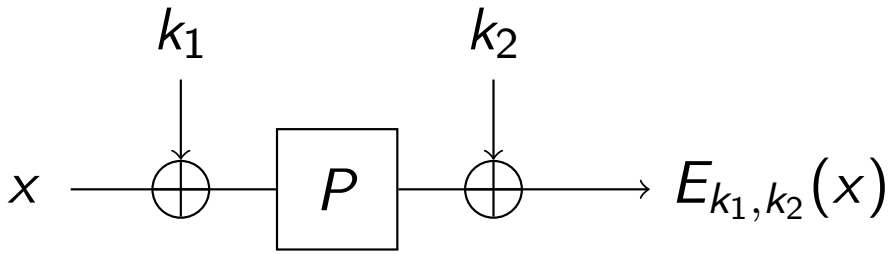
Create a periodic function from a cipher, whose period is a secret.

Characteristics

- Polynomial time, only $\mathcal{O}(n)$ queries
- Require quantum queries

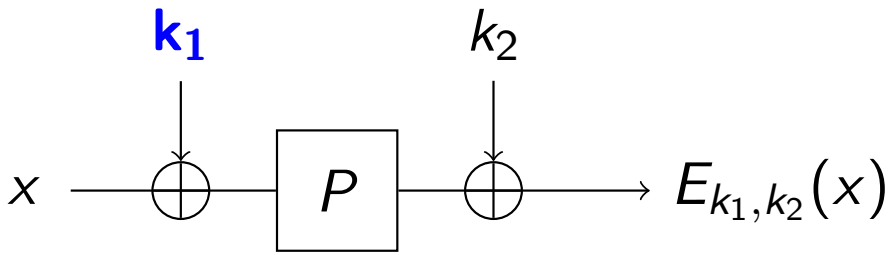
The Even-Mansour Cipher

Built from a random permutation $P : \{0,1\}^{\mathbf{n}} \to \{0,1\}^{\mathbf{n}}$.



$$E_{k_1,k_2}(x) = k_2 \oplus P(x \oplus k_1)$$

Classical collision attack



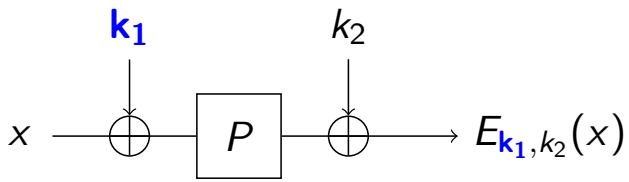
Let $f(x) = E_{\mathbf{k_1},k_2}(x) \oplus P(x) = k_2 \oplus P(x \oplus \mathbf{k_1}) \oplus P(x)$

It satisfies $f(x \oplus \mathbf{k_1}) = f(x)$: a collision of f yields $\mathbf{k_1}$ w.h.p.

Attack Cost

- Time $2^{\mathbf{n}/2}$, memory 1
- Data $2^{\mathbf{n}/2}$

Classical data-time tradeoff



Let $g(y) = P(y) \oplus P(y \oplus \mathbf{1})$, then

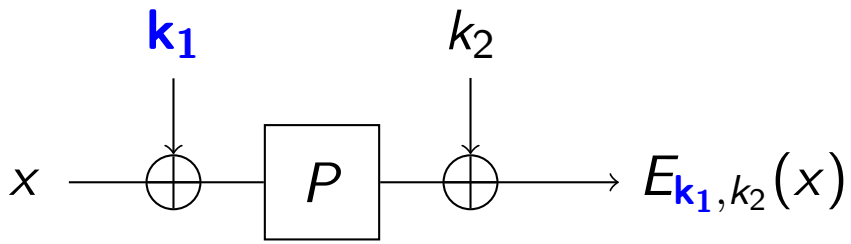
$$g(x \oplus \mathbf{k_1}) = E_{\mathbf{k_1},k_2}(x) \oplus E_{\mathbf{k_1},k_2}(x \oplus \mathbf{1})$$

Attack

- Collect D values of $E_{\mathbf{k_1},k_2}(x) \oplus E_{\mathbf{k_1},k_2}(x \oplus \mathbf{1})$
- Store a database $\mathcal{D}$ of size D
- Find y such that $g(y) \in \mathcal{D}$, in time $2^{\mathbf{n}}/D$
- With good probability $y = x \oplus \mathbf{k_1}$

Trade-off curve: $T \cdot D = 2^{\mathbf{n}}$, uses D memory.

Quantum attack [KM12]



Quantum attack

$f(x) = E_{\mathbf{k_1},k_2}(x) \oplus P(x)$ satisfies $f(x \oplus \mathbf{k_1}) = f(x)$.

Even-Mansour is broken in polynomial time in the Q2 model.

Simon-based cryptanalysis

- Distinguishers on Feistel constructions
- Multiple quantum slide attacks
- AEZ
- Multiple modes of operation
- Quantum related-key attacks
- ...

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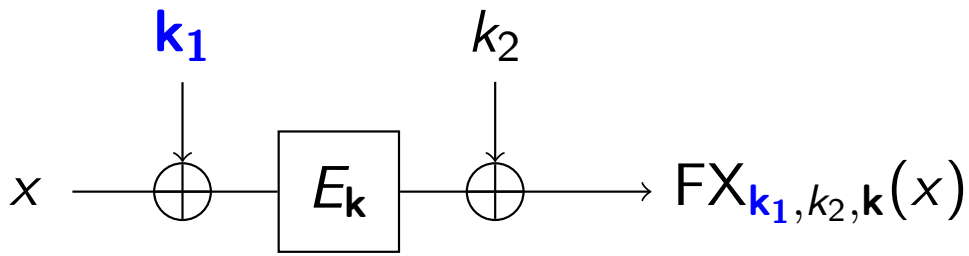
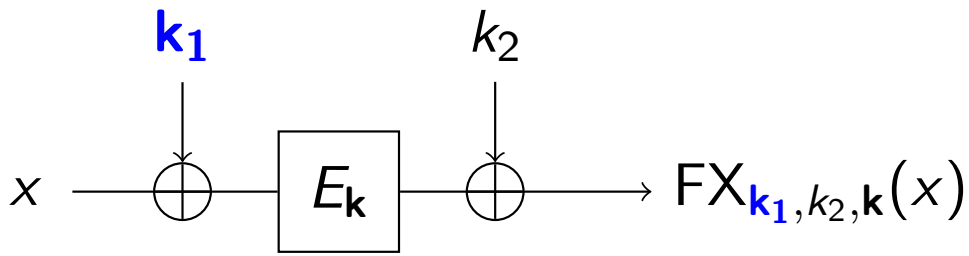
Require quantum queries

The Offline Simon's Algorithm

Example: FX construction



Example: FX construction



Quantum attack: "Grover-meet-Simon" [LM17]

- Quantum search for $\mathbf{k}$
- Checking: Kuwakado and Morii's attack works $\iff$ the guess of $\mathbf{k}$ is correct

Total time is $\underbrace{poly(\mathbf{n})}_{\text{Simon's algo}} \times \underbrace{2^{|\mathbf{k}|/2}}_{\text{Grover's iterates}}$.

Our remark on FX [BHNSS19]

The function:

$$f_z(x) = \mathrm{FX}_{\mathbf{k_1},k_2,\mathbf{k}}(x) \oplus E_z(x)$$

has $f_z(x \oplus \mathbf{k_1}) = f_z(x)$ **if** $z = \mathbf{k}$ **(the good one)**.
f_z is a sum:

$$f_z(x) = \underbrace{\mathrm{FX}_{\mathbf{k_1},k_2,\mathbf{k}}(x)}_{\substack{\textbf{Independent of z: online function } f}} \oplus \underbrace{E_z(x)}_{\substack{\textbf{Grover search space: offline function } g}}$$

For one query to f_z

- Do one quantum query to $\mathrm{FX}_{\mathbf{k_1},k_2,\mathbf{k}}(x)$ (fixed!)
- Add $E_z(x)$ (only depends on public information)

A new test algorithm

1. Begin with $\mathcal{O}(\mathbf{n})$ states of the form $\sum_{x\in\{0,1\}^{\mathbf{n}}} |x\rangle\,|f(x)\rangle$
2. Make queries to g and build: $\sum_{x\in\{0,1\}^{\mathbf{n}}} |x\rangle\,|(f\oplus g)(x)\rangle$
3. With Simon's algorithm, obtain a single output bit: whether $f\oplus g$ has a period or not
4. **Revert the computations**, query g again, put the "sample states" back to

$$\sum_{x\in\{0,1\}^{\mathbf{n}}} |x\rangle\,|f(x)\rangle$$

This emulates a reversible quantum circuit that tests for the periodicity of $f\oplus g$, *with only preprocessed queries to* f.

Our Q2 attack on FX

The queries to $\mathrm{FX}_{\mathbf{k_1},k_2,\mathbf{k}}(x)$ are made beforehand.

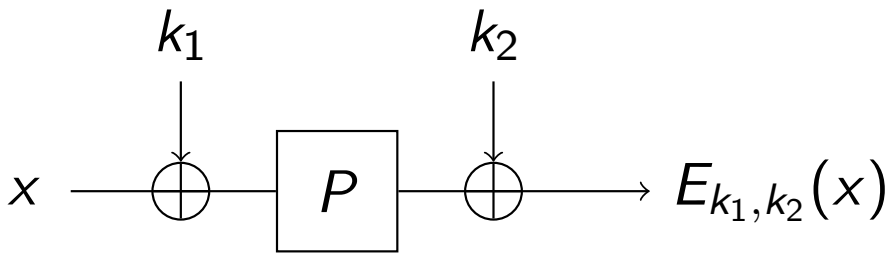
Test function

- Fetch the sample states $\sum_{x\in\{0,1\}^{\mathbf{n}}} |x\rangle\,|\mathrm{FX}_{\mathbf{k_1},k_2,\mathbf{k}}(x)\rangle$
- Create the Simon states $\sum_{x\in\{0,1\}^{\mathbf{n}}} |x\rangle\,|\mathrm{FX}_{\mathbf{k_1},k_2,\mathbf{k}}(x)\oplus E_z(x)\rangle$
- Test if there is a period
- **Revert** the operations and get back the sample states

Quantum search cost

- Time unchanged
- Queries reduced from $\mathcal{O}\left(\mathbf{n}2^{|\mathbf{k}|/2}\right)$ to $\mathcal{O}\left(\mathbf{n}\right)$
- Needs $\mathcal{O}\left(\mathbf{n}^2\right)$ Qubits

Back to the Even-Mansour cipher

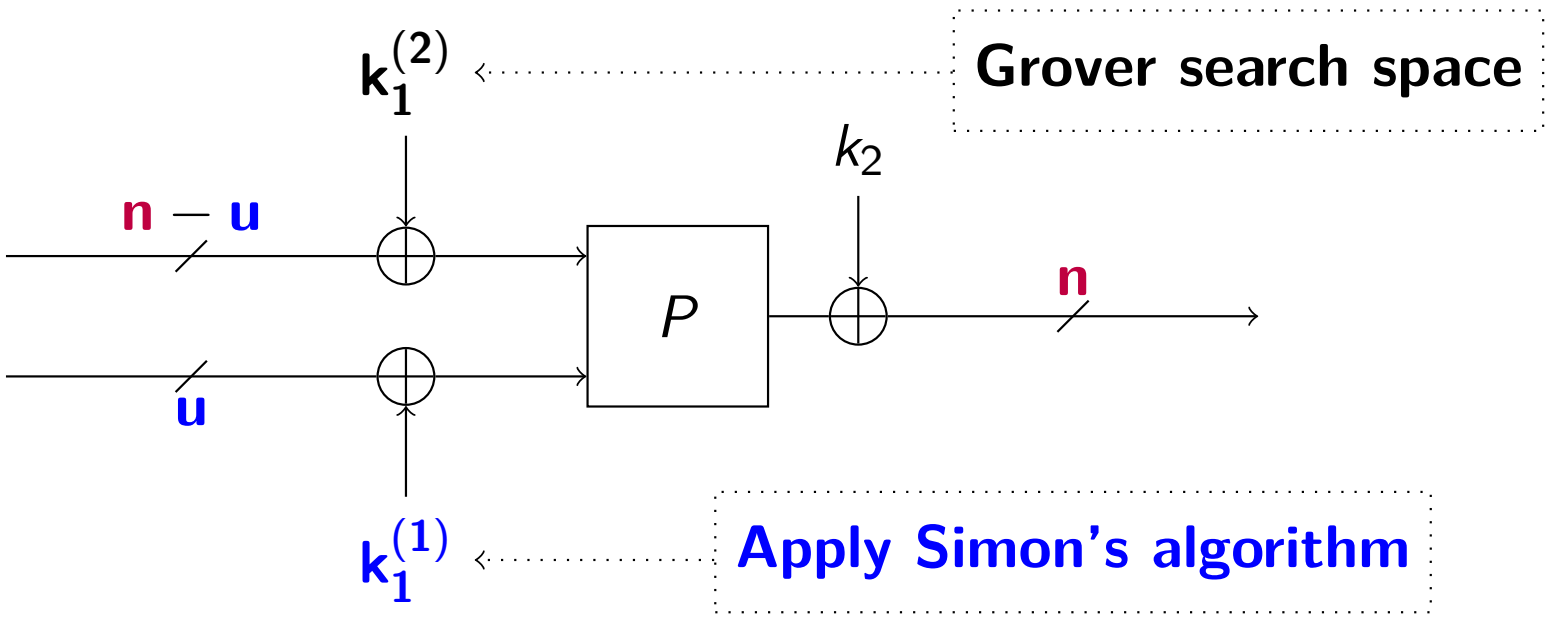


Producing the sample states with Q1 queries is possible. . . in time $2^{\mathbf{n}}$, with the whole codebook.

$\Longrightarrow$ not an attack.

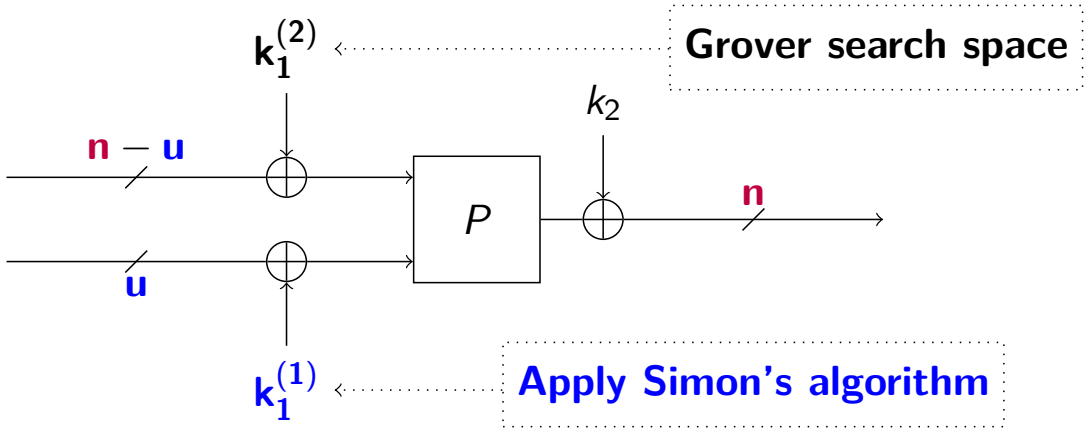
Q1 attack on Even-Mansour

We separate k_1 in two parts.



Define $f(x) = E_{k_1,k_2}(x \| 0^{\mathbf{n}-\mathbf{u}}) \oplus P(x \| \mathbf{k_1^{(2)}})$.

Q1 attack on Even-Mansour (ctd.)



$f(x) = E_{k_1,k_2}(x\|0^{\mathbf{n}-\mathbf{u}}) \oplus P(x\|\mathbf{k}_1^{(2)})$ has period $\mathbf{k}_1^{(1)}$

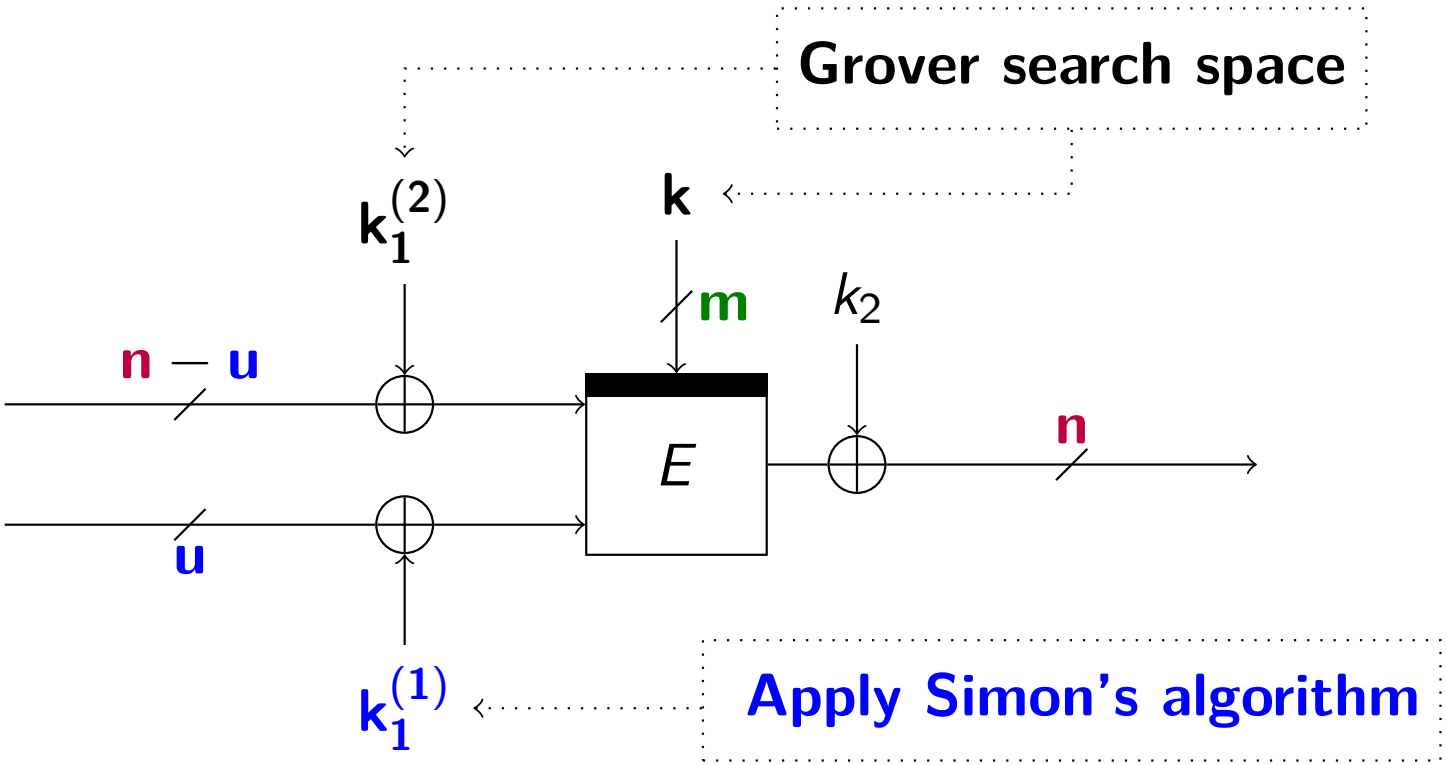
1. Produce the sample states $\sum_x |x\rangle |E_{k_1,k_2}(x\|0^{\mathbf{n}-\mathbf{u}})\rangle$
2. Search the good $\mathbf{k}_1^{(2)}$ ($\mathbf{n}-\mathbf{u}$ bits)

Data: $2^{\mathbf{u}}$ **Memory:** $\mathcal{O}(\mathbf{nu})$

Time: $2^{\mathbf{u}} + 2^{(\mathbf{n}-\mathbf{u})/2}$

Balances when Data = Time = $2^{\mathbf{n}/3}$

Q1 attack on the FX construction



We do the same, with more guesses in Grover's algorithm:
Data = Time = $2^{(n+m)/3}$.

Summary

The offline approach

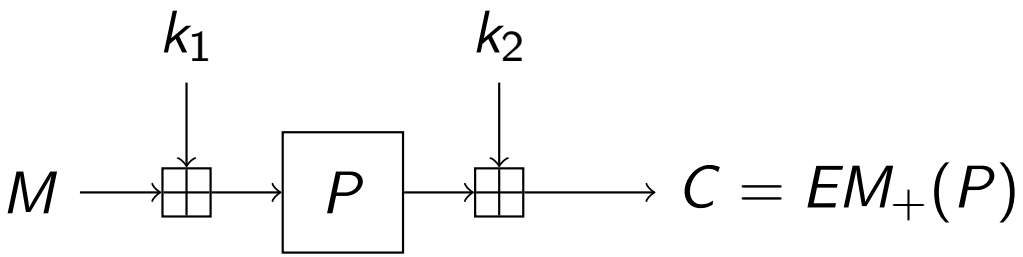
We reuse the quantum queries for each iteration of Simon's algorithm when the periodic function allows it.

Consequences

- Drastically reduces the number of quantum queries.
- Allows to convert a Q2 attack into a Q1 attack.
- Provides the best known Q1 attacks

Hidden shift algorithms

Avoiding Simon's attacks [AR17]



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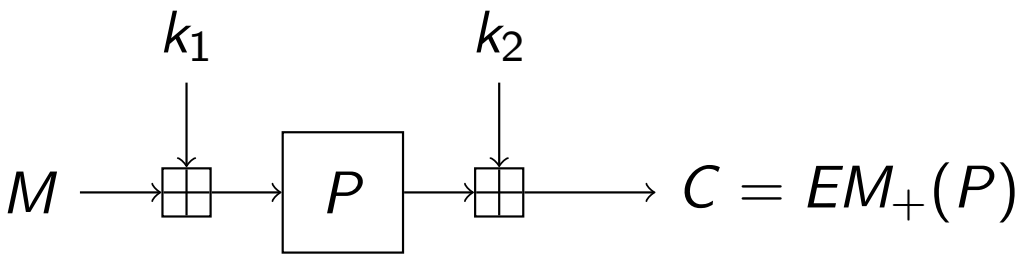

Properties

- $f(x) = EM_+(x) - EM_+(x+1) \quad g(x) = P(x) - P(x+1)$
- $f(x) = g(x + k_1)$

Security

- No (known) polynomial algorithm
- Hidden shift algorithm!

Hidden shift problem

Hidden shift problem

- $f, g : \mathcal{G} \to X$ injective
- $s \in \mathcal{G}$
- $f(x) = g(x + s)$
- Goal : find s, given oracle access to f and g.

Classical resolution

Find a collision, in $\Omega\left(2^{n/2}\right)$ samples.

Easy cases

- $\mathcal{G} = (\mathbb{Z}/(2))^n$: Simon's algorithm
- $f = g$: Shor's algorithm

Labeled qubits (in $\mathbb{Z}/(2^n)$)

- *Start with* $|0\rangle\,|0\rangle\,|0\rangle$

Labeled qubits (in $\mathbb{Z}/(2^n)$)

- Start with $|0\rangle |0\rangle |0\rangle$
- *Apply H* : $\sum_{b=0}^{1}, \sum_{x=0}^{2^n-1} |b\rangle |x\rangle |0\rangle$

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- *Apply the quantum oracles*

$$\sum_x |0\rangle |x\rangle |f(x)\rangle + |1\rangle |x\rangle |g(x)\rangle$$

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$$\sum_x |0\rangle |x\rangle |f(x)\rangle + |1\rangle |x\rangle |g(x)\rangle$$

- *Measure in the last register* $y = f(x_0) = g(x_0 + s)$

$$|0\rangle |x_0\rangle + |1\rangle |x_0 + s\rangle$$

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$$|0\rangle |x_0\rangle + |1\rangle |x_0 + s\rangle$$

- *Apply a quantum Fourier Transform*

$$\sum_\ell \exp\left(2i\pi \frac{x_0 \ell}{2^n}\right) |0\rangle |\ell\rangle + \exp\left(2i\pi \frac{(x_0 + s)\ell}{2^n}\right) |1\rangle |\ell\rangle$$

Labeled qubits (in $\mathbb{Z}/(2^n)$)

- Start with $|0\rangle|0\rangle|0\rangle$
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- *Measure* ℓ

$$|\psi_\ell\rangle = |0\rangle + \exp\left(2i\pi\frac{s\ell}{2^n}\right)|1\rangle$$

Combining qubits

Labeled qubits

- $|\psi_\ell\rangle = |0\rangle + \exp\left(2i\pi\frac{s\ell}{2^n}\right)|1\rangle$
- $|\psi_{2^n-1}\rangle = |0\rangle + (-1)^s|1\rangle$

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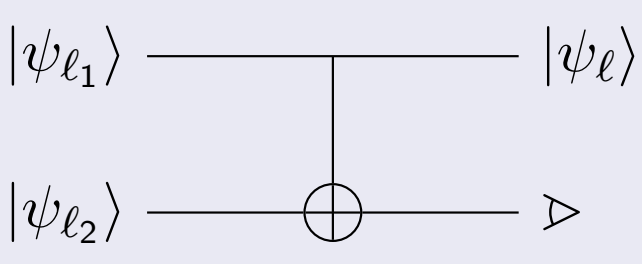
Combination: CNOT [Kup05]

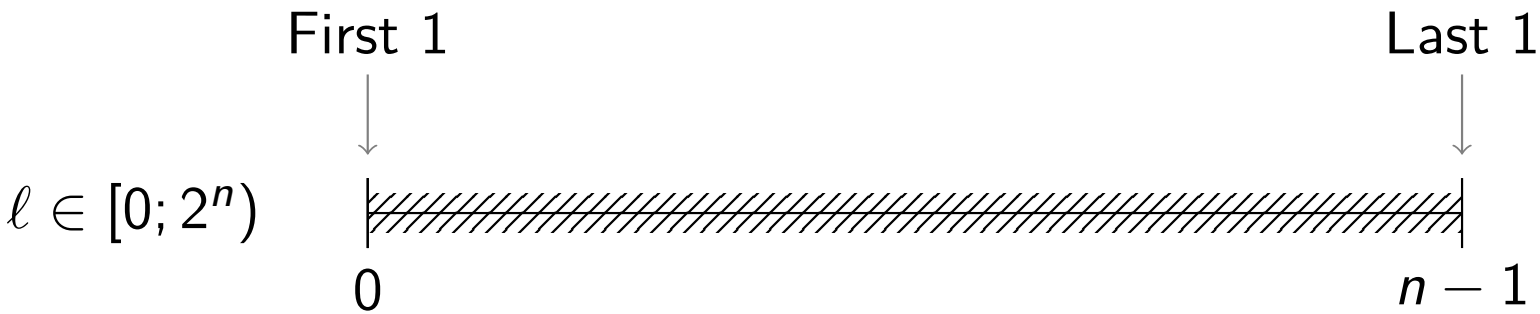


$$(\ell_1, \ell_2) \mapsto \ell_1 \pm \ell_2 \mod 2^n$$

Kuperberg's first algorithm [Kup05]

Look for partial collisions

Find ℓ_1, ℓ_2: $\ell_1 \pm \ell_2 = 2^a \ell'$



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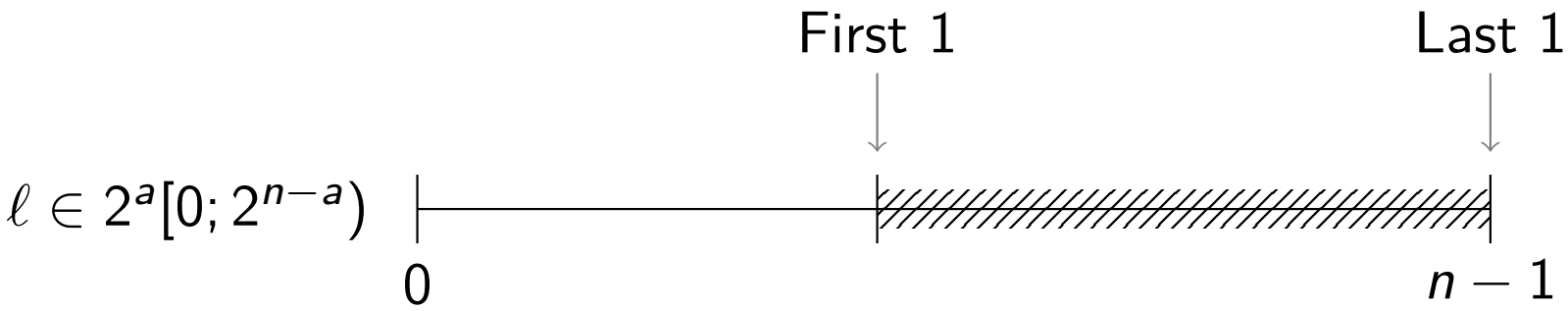

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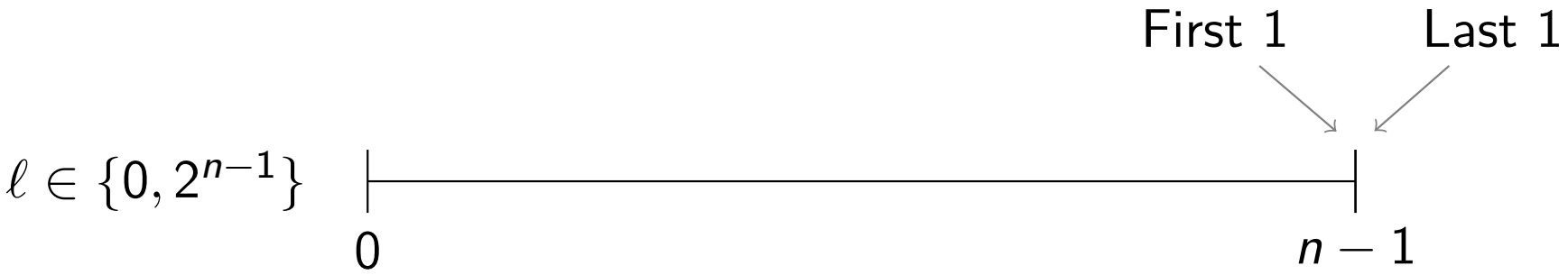
Look for partial collisions

Find ℓ_1, ℓ_2: $\ell_1 \pm \ell_2 = 2^a \ell'$



Summary

Algorithms principles

- Produce random $|\psi_\ell\rangle = |0\rangle + \exp\left(2i\pi\frac{s\ell}{2^n}\right)|1\rangle$
- Combine them to converge to $|\psi_{2^{n-1}}\rangle = |0\rangle + (-1)^s|1\rangle$

Complexity

Asymptotic complexity $\widetilde{\mathcal{O}}\left(2^{\sqrt{2\log_2(3)n}}\right)$ quantum time and memory

New results

Algorithm Improvements

More efficient recovery of the other bits of s, gain $\mathcal{O}(\mathbf{n})$.

Simulations

Heuristic complexity of $0.7 \times 2^{1.8\sqrt{n}}$ quantum time and memory.

Simon-meets-Kuperberg

More efficient algorithm for $(\mathbb{Z}/(2^w))^p$

Regev's variant [Reg04]

Same principles, different combination method.

- *Start with* $|\psi_{\ell_1}\rangle, \ldots |\psi_{\ell_k}\rangle$

$$\bigotimes_j |\psi_{\ell_j}\rangle = \sum_{b_j \in \{0,1\}} \exp\left(2i\pi \frac{s}{2^n}\left(\sum \ell_j b_j\right)\right) |b_1\rangle \ldots |b_k\rangle$$

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- *Compute* $\sum_j \ell_j b_j \mod 2^m$

$$\sum_{b_j \in \{0,1\}} \exp\left(2i\pi \frac{s}{2^n}\left(\sum \ell_j b_j\right)\right) |b_1\rangle \ldots |b_k\rangle \left|\sum \ell_j b_j \mod 2^m\right\rangle$$

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- *Measure a value V*

$$\sum_{b_j \in \{0,1\}, \sum_j \ell_j b_j \mod 2^m = V} \exp\left(2i\pi \frac{s}{2^n}\left(\sum \ell_j b_j\right)\right) |b_1\rangle \ldots |b_k\rangle$$

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- *Find two solutions* $(b_1, \ldots, b_k), (b'_1, \ldots, b'_k)$ *of*

$$\sum_j \ell_j b_j \mod 2^m = V$$

Regev's variant [Reg04]

- Measure a value V

$$\sum_{b_j \in \{0,1\}, \sum_j \ell_j b_j \mod 2^m = V} \exp\left(2i\pi \frac{s}{2^n}\left(\sum \ell_j b_j\right)\right) |b_1\rangle \ldots |b_k\rangle$$

- Find two solutions $(b_1, \ldots, b_k), (b'_1, \ldots, b'_k)$ of

$$\sum_j \ell_j b_j \mod 2^m = V$$

- *Project on them*

Regev's variant [Reg04]

- Measure a value V

$$\sum_{b_j \in \{0,1\}, \sum_j \ell_j b_j \mod 2^m = V} \exp\left(2i\pi \frac{s}{2^n}\left(\sum \ell_j b_j\right)\right) |b_1\rangle \ldots |b_k\rangle$$

- Find two solutions $(b_1, \ldots, b_k), (b'_1, \ldots, b'_k)$ of

$$\sum_j \ell_j b_j \mod 2^m = V$$

- Project on them
- *Map* $(b_1, \ldots, b_k)$ *to* 0, $(b'_1, \ldots, b'_k)$ *to 1*

$$|0\rangle + \exp\left(2i\pi \frac{s}{2^n}\left(\sum \ell_j b'_j - \sum \ell_j b_j\right)\right) |1\rangle$$

Regev's variant [Reg04]

- Measure a value V

$$\sum_{b_j \in \{0,1\}, \sum_j \ell_j b_j \mod 2^m = V} \exp\left(2i\pi \frac{s}{2^n}\left(\sum \ell_j b_j\right)\right) |b_1\rangle \ldots |b_k\rangle$$

- Find two solutions $(b_1, \ldots, b_k), (b'_1, \ldots, b'_k)$ of

$$\sum_j \ell_j b_j \mod 2^m = V$$

- Project on them
- Map $(b_1, \ldots, b_k)$ to 0, $(b'_1, \ldots, b'_k)$ to 1

$$|0\rangle + \exp\left(2i\pi \frac{s}{2^n}\left(\sum \ell_j b'_j - \sum \ell_j b_j\right)\right) |1\rangle$$

- *New labeled qubit* $|\psi_\ell\rangle$, *with* $\ell = \sum \ell_j b'_j - \sum \ell_j b_j$, $2^m | \ell$

Pipeline of routines

Routine

- Input: $|\psi_{\ell_1}\rangle, \ldots |\psi_{\ell_k}\rangle$, m
- Output: $|\psi_\ell\rangle$, $2^m|\ell$
- Needs to compute the solutions of $\sum_j \ell_j b_j \mod 2^m = V$

Algorithm cost [CJS14]

- $\widetilde{\mathcal{O}}\left(2^{\sqrt{2\mathbf{n}\log_2(\mathbf{n})}}\right)$ quantum time and queries
- $\mathcal{O}(\mathbf{n})$ quantum memory

New tradeoffs

Algorithm cost

- Number of queries: increases with $\mathbf{n}/k$
- Classical cost: increases with k

$$L(\alpha, c) = 2^{(c+o(1))\mathbf{n}^{\alpha}\log(\mathbf{n})^{1-\alpha}}$$

Subset-sum algorithm of dimension k in 2^{bk}

Tradeoffs

- Minimal classical cost: $L(1/2, \sqrt{b})$
- Tradeoff classical $L(1-\alpha, *)$, quantum $L(\alpha, *)$ $(\alpha < 1/2)$
- Minimal quantum cost: $\mathcal{O}\left(\mathbf{n}^{\alpha+1}\right)$, with classical $\widetilde{\mathcal{O}}\left(2^{b\mathbf{n}/\alpha}\right)$ $(\alpha \geq 1)$

Summary

Hidden shift problem

$f, g : \mathcal{G} \to X$, $f(x) = g(x + s)$

Many tradeoffs

- Most time-efficient: $2^{\sqrt{2\mathbf{n}}}$ [Kup13]
- Can trade between classical and quantum time, classical memory, quantum queries
- Even a small quantum computer reduces the cost

Special cases

- $\mathcal{G} = (\mathbb{Z}/(2))^n$: Simon's algorithm
- Intermediate case $\mathcal{G} = (\mathbb{Z}/(2^w))^p$
- $f = g$: Shor's algorithm

Hidden shift Cryptanalysis

General idea

Create from a cipher two functions that fullfill $f(x) = g(x + s)$.

Applications

- Ciphers with modular addition instead of xors
- Polynomial MACs
- Offline variant possible

Numerical estimates for Even-Mansour-like constructions

- $\mathbf{n} = 128$, 16 bits of security
- $\mathbf{n} = 1600$, 56 bits of security

Key exchanges

Diffie-Hellman Key Exchange

Protocol

$$G, g \in G$$

(α) Alice

Bob (β)

Diffie-Hellman Key Exchange

Protocol

$$G, g \in G$$

$$(\alpha) \text{ Alice} \xrightarrow{[\alpha]g} \text{Bob } (\beta)$$

Diffie-Hellman Key Exchange

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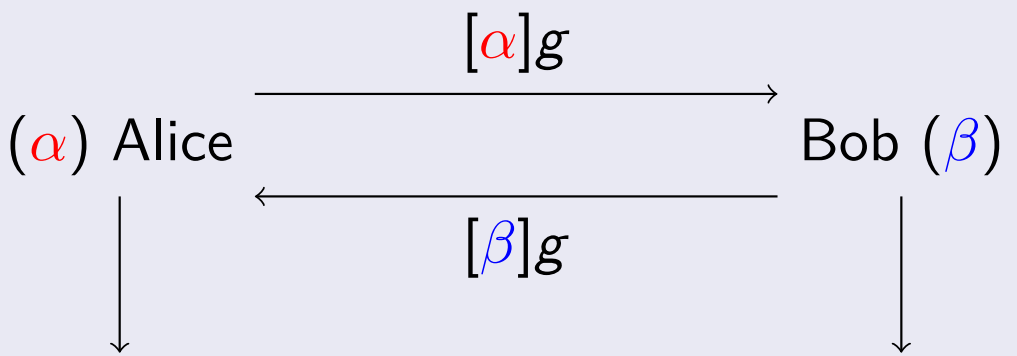

Diffie-Hellman Key Exchange

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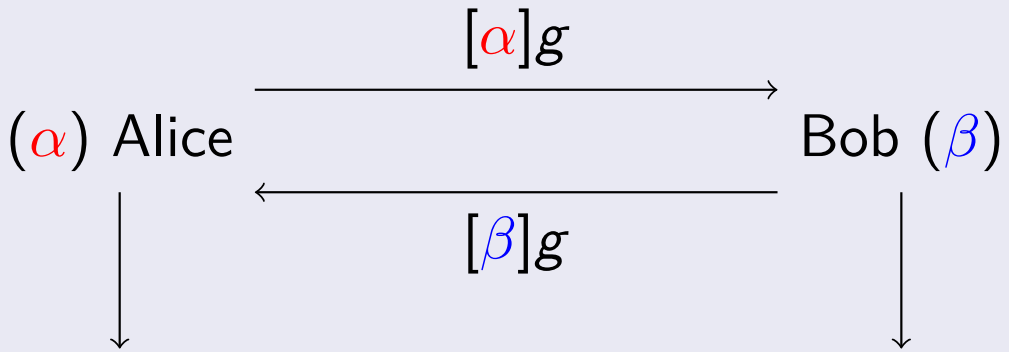

Security

$$(g, [\alpha]g) \to \alpha$$

Quantum Security of Diffie-Hellman

Discrete log

$$(g, [\alpha]g) \to \alpha$$

Hidden subgroup

- $f(x, y) = [x]g + [y][\alpha]g$
- $[\alpha]g = [1]([\alpha]g)$
- $f(x, y) = f(x + \alpha, y - 1)$

Broken with Shor's algorithm.

Shor's algorithm [Sho94]

Shor's problem

- $f : \mathbb{Z} \to \mathbb{Z}$
- $r \in \mathbb{Z}$
- $\forall x, f(x) = f(x + r)$
- f hides the period r
- $r < 2^n$
- Goal : find r, given oracle access to f.

Shor's circuit ($f(x) = f(x + r)$)

Quantum circuit

- *Start from* $|0\rangle\,|0\rangle$

Measure biaised toward ℓ such that $\frac{r\ell}{2^n}$ is close to an integer.

Shor's circuit ($f(x) = f(x + r)$)

Quantum circuit

- Start from $|0\rangle |0\rangle$
- *Apply H* : $\sum_x |x\rangle |0\rangle$

Measure biaised toward ℓ such that $\frac{r\ell}{2^n}$ is close to an integer.

Shor's circuit ($f(x) = f(x + r)$)

Quantum circuit

- Start from $|0\rangle |0\rangle$
- Apply H : $\sum_x |x\rangle |0\rangle$
- *Apply the oracle* $\sum_x |x\rangle |f(x)\rangle$

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Quantum circuit

- Start from $|0\rangle |0\rangle$
- Apply H : $\sum_x |x\rangle |0\rangle$
- Apply the oracle $\sum_x |x\rangle |f(x)\rangle$
- *Measure the second register* $\sum_{j=0}^{\alpha} |x_0 + jr\rangle$

Measure biaised toward ℓ such that $\frac{r\ell}{2^n}$ is close to an integer.

Shor's circuit ($f(x) = f(x+r)$)

Quantum circuit

- Start from $|0\rangle|0\rangle$
- Apply H : $\sum_x |x\rangle|0\rangle$
- Apply the oracle $\sum_x |x\rangle|f(x)\rangle$
- Measure the second register $\sum_{j=0}^{\alpha} |x_0 + jr\rangle$
- *Apply a QFT* $\sum_{j=0}^{\alpha} \sum_{\ell} \exp\left(2i\pi \frac{(x_0+jr)\ell}{2^n}\right) |\ell\rangle$

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Shor's circuit ($f(x) = f(x+r)$)

Quantum circuit

- Start from $|0\rangle|0\rangle$
- Apply H : $\sum_x |x\rangle|0\rangle$
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- Measure the second register $\sum_{j=0}^{\alpha} |x_0 + jr\rangle$
- Apply a QFT $\sum_{j=0}^{\alpha} \sum_\ell \exp\left(2i\pi \frac{(x_0+jr)\ell}{2^n}\right)|\ell\rangle$
- *Amplitude of ℓ :*

$$\sum_{j=0}^{\alpha} \exp\left(2i\pi \frac{jr\ell}{2^n}\right)$$

Measure biaised toward ℓ such that $\frac{r\ell}{2^n}$ is close to an integer.

Shor's algorithm

Shor's problem

Given f such that $f(x) = f(x + r)$, find r.

Summary

- Sample biaised values
- Recover r in $O(1)$ queries
- Generalizes to any abelian group

Applications

Breaks discrete log and factoring

A Dillema

The problem

- Commutation allows key exchange
- Commutative groups are too easy

A Dillema

The problem

- Commutation allows key exchange
- Commutative groups are too easy

Commutative group action

- G a commutative group
- X a set
- $* : G \times X \to X$
- $g * (g' * x) = (gg') * x$

Hard homogeneous space [Cou96]

- Given g, x, computing $g * x$ is easy
- Given x, $g * x$, computing g is hard

Group action Diffie-Hellman

Protocol

$$G, X, x \in X$$

(α) Alice

Bob (β)

Examples of schemes

- Ordinary isogeny-based schemes
- CSIDH
- But not SIKE

Group action Diffie-Hellman

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$$(\alpha)\ \text{Alice} \xrightarrow{\alpha * x} \text{Bob}\ (\beta)$$

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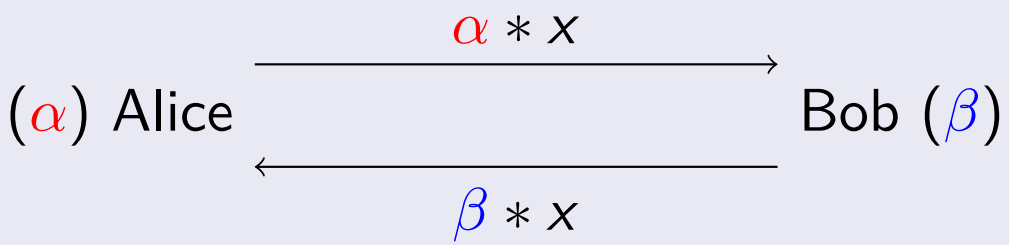

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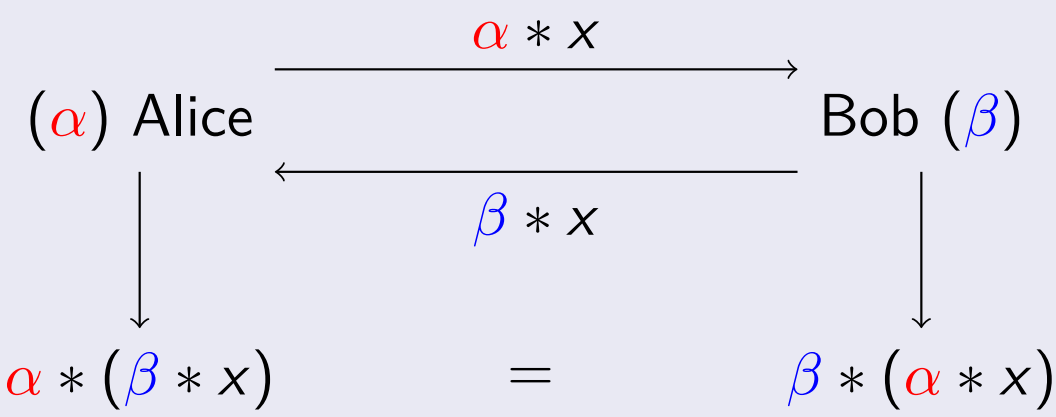

Examples of schemes

- Ordinary isogeny-based schemes
- CSIDH
- But not SIKE

Attacking the commutative group action

Inverting the group action

$$(x, \alpha * x) \to \alpha$$

A Hidden Shift!

- $f(z) = z * x$
- $g(z) = z * (\alpha * x)$
- $f(z + \alpha) = g(z)$

Numerical estimates for CSIDH

Claimed security

For $p \sim 512$ bits: $\simeq 2^{81}$ quantum operations, 2^{128} classical operations

A tradeoff

Attack in 2^{67} quantum operations, 2^{86} classical memory and operations

Safe instances

$p \sim 2000$ bits: 2^{80} quantum operations, 2^{78} classical operations, 2^{49} classical memory

Conclusion

New quantum algorithms

- The offline approach
- Improvements and tradeoffs for hidden shift algorithms

Many cryptanalyses

- Many instances of hidden structure scenarios
- Hidden structures are exploitable *even with classical queries*