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INTRODUCTION TO CODING THEORY

- An **$[n, k]$ linear code $\mathcal{C}$** over $\mathbb{F}_q$ is a k-dimensional subspace of $\mathbb{F}_q^n$.

  Its **size** is $M = q^k$, the **information rate** is $R = \frac{k}{n}$ and the **redundancy** is $n - k$.
- The **generator matrix** of $\mathcal{C}$ is a $k \times n$ matrix G whose rows form a basis of $\mathcal{C}$, i.e.

$$\mathcal{C} = \left\{ \mathbf{x}G \mid \mathbf{x} \in \mathbb{F}_q^k \right\}.$$

- The **parity-check matrix** of $\mathcal{C}$ is an $(n-k) \times n$ matrix H whose nullspace is generated by the codewords of $\mathcal{C}$, i.e.

$$\mathcal{C} = \left\{ \mathbf{y} \in \mathbb{F}_q^n \mid H\mathbf{y}^T = 0 \right\}.$$

- The **hamming distance** between $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$ is $d_H(\mathbf{x}, \mathbf{y}) = |\{i \mid x_i \neq y_i\}|$.
- The **minimum distance** of $\mathcal{C}$ is

$$d(\mathcal{C}) = \min \{ d_H(\mathbf{c}_1, \mathbf{c}_2) \mid \mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C} \text{ and } \mathbf{c}_1 \neq \mathbf{c}_2 \}.$$

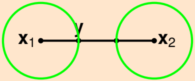

FIGURE: If $d(\mathcal{C}) = 3$

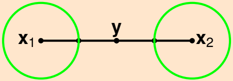

FIGURE: If $d(\mathcal{C}) = 4$

Decoding Linear Codes

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The Decoding problem:

Input: a Generator matrix $G \in \mathbb{F}_q^{k\times n}$ of $\mathcal{C}$ and the received word $\mathbf{y} \in \mathbb{F}_q^n$

Output: A closest codeword $\mathbf{c}$, i.e.

$$\mathbf{c} \in \mathcal{C} \;:\; d_H(\mathbf{c}, \mathbf{y}) = \min\{d_H(\hat{\mathbf{c}}, \mathbf{y}) \mid \hat{\mathbf{c}} \in \mathcal{C}\}$$

Decoding **arbitrary linear codes**: Exponential complexity

Decoding Special Classes of Codes

Efficient decoding algorithms up to half the minimum distance for:

1. Generalized Reed-Solomon codes
2. Goppa codes
3. Algebraic Geometry codes

Polynomial complexity $\sim O\left(n^3\right)$

- Peterson, Arimoto, 1960
- Berlekamp-Massy, 1963
- Justensen-Larsen-Havemose-Jensen-Høholdt, 1989
- Skorobogatov-Vladut, 1990
- Sakata, 1990
- Feng-Rao, Duursma 1993
- Sudam, Guruswami, 1997

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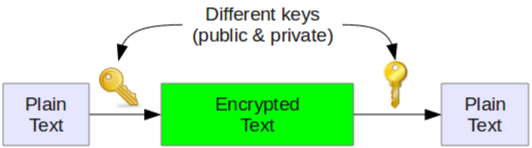
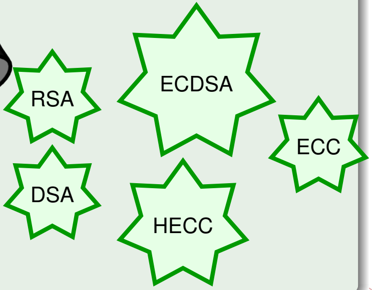
Public-Key Cryptosystems



Most PKC are based on Number-theoretic problems

➜ It can be attacked in polynomial time using **Shor's algorithm**



McEliece cryptosystem

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➜ McEliece introduced the first PKC based on Error-Correcting Codes in 1978.

Advantages:

1. Fast encryption (matrix-vector multiplication) and decryption functions.
2. Interesting candidate for post-quantum cryptography.

Drawback:

➢ Large key size.

R. J. McEliece.
A public-key cryptosystem based on algebraic coding theory.
DSN Progress Report, 42-44:114-116, 1978.

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McEliece Cryptosystem

➜ $t \in \mathbb{N}^* \implies$ **Error-correcting capacity** of $\mathcal{C}$

Consider any triplet:

$$\left(\mathcal{C}, \mathcal{A}_{\mathcal{C}}(t) \right)$$

➜ $[n, k]_q$ **linear code** with an efficient decoding algorithm

- Let G be a non structured generator matrix of $\mathcal{C}$.

➜ ***"Efficient" decoding algorithm*** for $\mathcal{C}$ which corrects up to t errors.

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McEliece Cryptosystem

Key Generation

Given:

1. **McEliece Public Key:** $\mathcal{K}_{pub} = (G, t)$
2. **McEliece Private Key:** $\mathcal{K}_{secret} = (\mathcal{A}_{\mathcal{C}})$

Encryption

Encrypt a message $\mathbf{m} \in \mathbb{F}_q^k$ as

$$\mathbf{y} = \mathbf{m}G + \mathbf{e}$$

where $\mathbf{e}$ is a random error vector of weight at most t.

Decryption

Using $\mathcal{K}_{secret}$, the receiver obtain $\mathbf{m}$.

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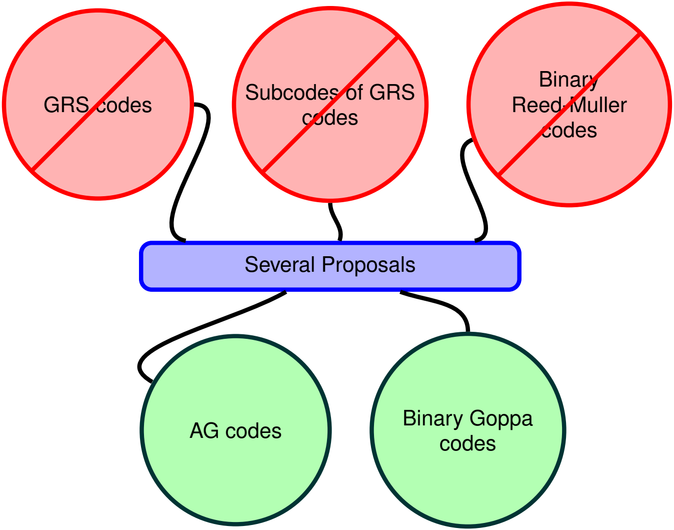

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GRS CODES

- The class of **GRS** codes was proposed by **Niederreiter** in 1986 for code-based PKC.

- ✖ **Sidelnikov-Shestakov** in 1992 introduced an algorithm that breaks this proposal in polynomial time.

Parameters	Key size	Security level
$[256, 128, 129]_{256}$	67 ko	2^{95}

Subcodes of GRS codes I

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- **Berger and Loidreau** in 2005 propose another version of the Niederreiter scheme designed to resist the Sidelnikov-Shestakov attack.
  - → **Main idea:** work with subcodes of the original GRS code.

- **Attacks:**
  - **Wieschebrink:** (2010)
    - Presents the first feasible attack to the Berger-Loidreau cryptosystem but is impractical for small subcodes.
    - Notes that if the square code of a subcode of a GRS code of parameters $[n, k]_q$ is itself a GRS code of dimension $2k - 1$ then we can apply Sidelnikov-Shestakov attack.
  - **M-Mártinez-Pellikaan:** (2012) Give a characterization of the possible parameters that should be used to avoid attacks on the Berger-Loidreau cryptosystem.

Subcodes of GRS codes II

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- **Wieschebrick** (2010) and **Baldi et al.** (2011) proposed other variants of the Niederreiter scheme.

- **Attacks: Couvreur et al.** (2013) provide a cryptanalysis of these schemes.

Binary Reed-Muller codes

- The class of **Binary Reed-Muller** codes was proposed by **Sidelnikov** in 1994 for code-based PKC.

- **Minder-Shokrollahi** in 2007 presents a sub-exponential time attack.

Parameters	Key size	Security level
$[1024, 176, 128]_2$	22.5 ko	2^{72}
$[2048, 232, 256]_2$	59, 4 ko	2^{93}

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- In 1996 **Janwa and Moreno** propose to use AG codes for the McEliece cryptosystem.

- This system was broken for:

1. Genus $g = 0$: by the **Sidelnikov-Shestakov** attack in 1992

   GRS codes are Algebraic Geometry codes on the projective line.

2. Genus $g = 1$: by **Minder-Shokrollahi** in 2007.
3. Genus $g \leq 2$: by **Faure-Minder** in 2008.
4. We can retrieve the **model of the curve** (in polynomial time) by **M-Martínez-Pellikaan-Ruano** in 2013

   $\Longrightarrow$ **It is NOT broken**

Parameters	Key size	Security level
$[171, 109, 61]_{128}$	16 ko	2^{66}

BINARY GOPPA CODES

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- The class of **binary goppa** codes was proposed by **McEliece** in 1977 for code-based PKC.

- ✔ McEliece with Goppa codes **has resisted cryptanalysis** so far!!

Parameters	Key size	Security level
$[1024, 524, 101]_2$	67 ko	2^{62}
$[2048, 1608, 48]_2$	412 ko	2^{96}

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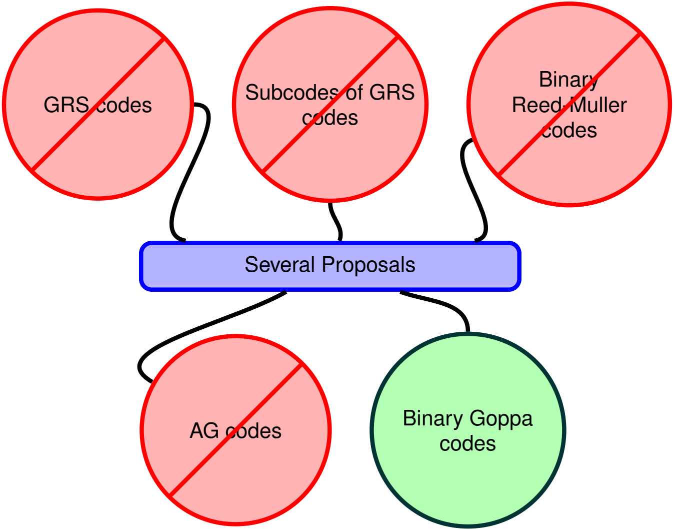

NOTATION

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➜ For all $\mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n$ we define:

- **Star Product:** $\mathbf{a} * \mathbf{b} = (a_1 b_1, \ldots, a_n b_n) \in \mathbb{F}_q^n$
- **Standard Inner Product:** $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{n} a_i b_i \in \mathbb{F}_q$

➜ For all subsets $A,\ B \subseteq \mathbb{F}_q^n$ we define:

- $A * B = \langle \{\mathbf{a} * \mathbf{b} \mid \mathbf{a} \in A \text{ and } \mathbf{b} \in B\} \rangle$

  For $B = A \Longrightarrow A * A$ is denoted as $A^{(2)}$

- $A \perp B \iff \langle \mathbf{a}, \mathbf{b} \rangle = 0 \quad \forall\, \mathbf{a} \in A \text{ and } \mathbf{b} \in B$

Decoding by Error Correcting Pairs

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Let $\mathcal{C}$ be a linear code. We denote by:

- $k(\mathcal{C})$ = dimension of $\mathcal{C}$
- $d(\mathcal{C})$ = minimum distance of $\mathcal{C}$

Error-correcting pairs (ECP)

Let $\mathcal{C}$ be an $\mathbb{F}_q$ linear code of length n. The pair (A, B) of $\mathbb{F}_q$-linear codes of length n is a t-ECP for $\mathcal{C}$ over if the following properties hold:

E.1 $(A * B) \perp \mathcal{C}$.
E.2 $k(A) > t$.
E.3 $d(B^{\perp}) > t$.
E.4 $d(A) + d(\mathcal{C}) > n$.

An $[n, k]_q$ code which has a t-ECP over $\mathbb{F}_q$ has a decoding algorithm with complexity $\mathcal{O}\left(n^{\mathbf{w}}\right)$.

Decoding by Error-Correcting Pairs (ECP) I

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Let:

- $\mathcal{C}$, A and B be linear subspaces of $\mathbb{F}_q^n$
- $\mathbf{y} \in \mathbb{F}_q^n$ be the received word with error vector $\mathbf{e}$

Compute:

$$K_{\mathbf{y}} = \{\mathbf{a} \in A \mid \langle \mathbf{y}, \mathbf{a} * \mathbf{b} \rangle = 0, \text{ for all } \mathbf{b} \in B\}$$

Remark: Condition 1

$$\text{If } A * B \subseteq \mathcal{C}^{\perp} \implies K_{\mathbf{y}} = K_{\mathbf{e}}$$

Let J be a subset of $\{1, \ldots, n\}$, define:

$$A(J) = \{\mathbf{a} \in A \mid a_j = 0, \text{ for all } j \in J\}$$

Lemma 1: Condition 3

$$\text{Let } I = \operatorname{supp}(\mathbf{e}) \text{ and } A * B \subseteq \mathcal{C}^{\perp}. \text{ If } d(B^{\perp}) > t \implies A(I) = K_{\mathbf{y}}$$

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Lemma 2: Condition 2

If $I = \mathrm{supp}(e)$ and $k(A) > t \implies \exists \mathbf{a} \in K_{\mathbf{y}} \setminus \{\mathbf{0}\}$

Lemma 3: Condition 4

Let $\mathbf{a} \in K_{\mathbf{y}} \setminus \{\mathbf{0}\}$ and define $J = \{j \mid a_j = 0\}$. Then:

1. If $d(B^{\perp}) > t$ then $I = \mathrm{supp}(\mathbf{e}) \subseteq J$
2. If $d(A) + d(\mathcal{C}) > n$ then there exists a unique solution to:

$$H\mathbf{x}^T = H\mathbf{y}^T \text{ such that } x_j \neq 0 \text{ for all } j \in J$$

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DECODING BY ERROR-CORRECTING PAIRS (ECP) III

1. Compute:

$$K_{\mathbf{y}} = \{\mathbf{a} \in A \mid \langle \mathbf{y}, \mathbf{a} * \mathbf{b} \rangle = 0, \text{ for all } \mathbf{b} \in B\}$$

> Find the zero space of a set of linear equations over $\mathbb{F}_q$

2. If $K_{\mathbf{y}} = \mathbf{0} \Longrightarrow$ **The received word has more than t errors**

➜ Else take a nonzero $\mathbf{a} \in K_{\mathbf{y}} = A(I)$ and define $J = \{j \mid a_j = 0\}$

3. Find $\mathbf{e} \in \mathbb{F}_q^n$ by solving the following linear equation (which has a **unique** solution):

$$H\mathbf{x}^T = H\mathbf{y}^T \quad \text{such that} \quad x_j \neq 0 \text{ for } j \in J$$

> Solve linear equations over $\mathbb{F}_q$

Complexity: $\sim \mathcal{O}\left(n^{\mathbf{w}}\right)$

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Generalized Reed-Solomon Codes I

Let

- $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-tuple of **mutually distinct** elements of $\mathbb{F}_q$.
- $\mathbf{b} = (b_1, \ldots, b_n)$ be an n-tuple of **nonzero** elements of $\mathbb{F}_q$.
- $k \in \mathbb{N} \ : \ k < n$

The **GRS code** $\mathrm{GRS}_k(\mathbf{a}, \mathbf{b})$ is defined by:

$$\mathrm{GRS}_k(\mathbf{a}, \mathbf{b}) = \{\mathbf{b} * f(\mathbf{a}) = (b_1 f(a_1), \ldots, b_n f(a_n)) \mid f \in \mathbb{F}_q[X]_{<k}\}$$

Generalized Reed-Solomon Codes II

Parameters of $\mathrm{GRS}_k(\mathbf{a}, \mathbf{b})$

The $\mathrm{GRS}_k(\mathbf{a}, \mathbf{b})$ is an **MDS** code with parameters $[n, k, n-k+1]_q$.

➜ A generator matrix of $\mathrm{GRS}_k(\mathbf{a}, \mathbf{b})$ is given by

$$G_{\mathbf{a},\mathbf{b}} = \begin{pmatrix} b_1 & \dots & b_n \\ b_1 a_1 & \dots & b_n a_n \\ \vdots & \ddots & \vdots \\ b_1 a_1^{k-1} & \dots & b_n a_n^{k-1} \end{pmatrix} \in \mathbb{F}_q^{k \times n}$$

Dual of a GRS code

The dual of a GRS code is again a GRS code. In particular:

$$\mathrm{GRS}_k(\mathbf{a}, \mathbf{b})^{\perp} = \mathrm{GRS}_{n-k}(\mathbf{a}, \mathbf{c}) \text{ for some } \mathbf{c} \text{ explicitly known}$$

➜ The $\mathrm{GRS}_k(\mathbf{a}, \mathbf{b})^{\perp}$ is an MDS code with parameters $[n, n-k, k+1]_q$.

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t-ECP for GRS I

Note that: $\mathrm{GRS}_k(\mathbf{a},\mathbf{b}) * \mathrm{GRS}_l(\mathbf{a},\mathbf{c}) = \mathrm{GRS}_{k+l-1}(\mathbf{a},\mathbf{b}*\mathbf{c})$

Let

$$A = \mathrm{GRS}_{t+1}(\mathbf{a},\mathbf{b}_1),\quad B = \mathrm{GRS}_t(\mathbf{a},\mathbf{b}_2) \text{ and}$$
$$\mathcal{C} = \mathrm{GRS}_{2t}(\mathbf{a},\mathbf{b}_1 * \mathbf{b}_2)^{\perp}$$

then (A, B) is a t-ECP for $\mathcal{C}$.

E.1 $A * B = \mathrm{GRS}_{2t}(\mathbf{a},\mathbf{b}_1 * \mathbf{b}_2) = \mathcal{C}^{\perp} \Rightarrow (A * B) \perp \mathcal{C}$
E.2 $k(A) > t$
E.3 $B^{\perp} = \mathrm{GRS}_{n-t}(\mathbf{a},\mathbf{c}_2) \Rightarrow d(B^{\perp}) = t + 1 > t$
E.4 $d(A) + d(\mathcal{C}) = (n - t) + (2t + 1) > n$

t-ECP FOR GRS II

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Conversely, let $\mathcal{C} = \mathrm{GRS}_{n-2t}(\mathbf{a}, \mathbf{b})$
then

$$A = \mathrm{GRS}_{t+1}(\mathbf{a}, \mathbf{c}) \text{ and } B = \mathrm{GRS}_t(\mathbf{a}, \mathbf{1})$$

is a t-ECP for $\mathcal{C}$ where $\mathbf{c} \in (\mathbb{F}_q \setminus \{0\})^n$ verifies that

$$\mathcal{C}^\perp = \mathrm{GRS}_{n-2t}(\mathbf{a}, \mathbf{b})^\perp = \mathrm{GRS}_{2t}(\mathbf{a}, \mathbf{c}).$$

Moreover an $[n, n-2t, 2t+1]_q$ code that has a t-ECP is a GRS code.

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Algebraic Geometry codes

- An AG code is defined by a triplet

$$\left(\mathcal{X}, \mathcal{P}, E\right)$$

- $\mathcal{X}$ is an algebraic curve of genus g over the finite field $\mathbb{F}_q$

Algebraic Curve = Smooth, Projective and Geometrically Connected Curve

Whose defining equations are polynomials with coefficients in $\mathbb{F}_q$.

- $\mathcal{P} = (P_1, \ldots, P_n)$ is an n-tuple of mutually distinct $\mathbb{F}_q$-rational points of $\mathcal{X}$

$D_{\mathcal{P}}$ denotes the divisor $D_{\mathcal{P}} = P_1 + \cdots + P_n$

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Algebraic Geometry codes

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➜ An AG code is defined by a triplet

$$\left(\mathcal{X}, \mathcal{P}, E\right)$$

➜ E is an $\mathbb{F}_q$-divisor of $\mathcal{X}$ such that

$$\mathrm{supp}(E) \cap \mathrm{supp}(D_{\mathcal{P}}) = \emptyset$$

ALGEBRAIC GEOMETRY CODES II

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DIVISORS ON CURVES

A **divisor** D on $\mathcal{X}$ is a formal finite sum:

$$D = \sum_{P \in \mathcal{X}} n_P P \text{ with } n_P \in \mathbb{Z} \text{ and } P \in \mathcal{X}$$

- If $n_P \geq 0$ for all $P \in \mathcal{X}$ then D is an **Effective Divisor**, $(D \geq 0)$.
- **Support of the divisor D:** $\mathrm{supp}(D) = \{P \mid n_P \neq 0\}$
- **Degree of the divisor D:** $\deg(D) = \sum_{P \in \mathcal{X}} n_p \deg(P)$

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Divisor of rational functions

The divisor of $f \in \mathbb{F}_q(\mathcal{X})$ is defined to be:

$$(f) = \underbrace{\sum_{P \text{ zero of } f} \mathrm{v}_P(f)P}_{(f)_0} - \underbrace{\sum_{P \text{ pole of } f} \mathrm{v}_P(f)P}_{(f)_\infty}$$

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Algebraic Geometry codes IV

Space of Rational Functions associated to the divisor E

$$L(E) = \{f \in \mathbb{F}_q(\mathcal{X}) \mid f = 0 \text{ or } (f) + E \geq 0\}$$

Intuitively: $f \in L(E) \iff f$ has enough zeros and not to many poles

Riemman-Roch Theorem

$$\dim(L(E)) \geq \deg(E) + 1 - g$$

Furthermore, if $\deg(E) > 2g - 2$ then $\dim(L(E)) = \deg(E) + 1 - g$

Algebraic Geometry codes V

➜ Let us consider the triplet:

$$\left(\mathcal{X}, \mathcal{P}, E\right)$$

➜ $\mathcal{X}$ is an algebraic curve of genus g over the finite field $\mathbb{F}_q$.

➜ $\mathcal{P}$ is an n-tuple of distinct $\mathbb{F}_q$-rational points of $\mathcal{X}$.

➜ E is an $\mathbb{F}_q$-divisor of $\mathcal{X}$ such that $\mathrm{supp}(E) \cap \mathrm{supp}(D_{\mathcal{P}}) = \emptyset$

Since $\mathrm{supp}(E) \cap \mathrm{supp}(D_{\mathcal{P}}) = \emptyset$ the following **evaluation map** is well defined:

$$\begin{array}{rccc} \mathrm{ev}_{\mathcal{P}}: & L(E) & \longrightarrow & \mathbb{F}_q^n \\ & f & \longmapsto & \mathrm{ev}_P(f) = (f(P_1), \ldots, f(P_n)) \end{array}$$

Algebraic Geometry Code (AG code)

The **AG code** associated to the triplet $(\mathcal{X}, \mathcal{P}, E)$ is:

$$\mathcal{C}_L(\mathcal{X}, \mathcal{P}, E) = \{\mathrm{ev}_{\mathcal{P}}(f) = (f(P_1), \ldots, f(P_n)) \mid f \in L(E)\}$$

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Algebraic Geometry codes VI

➜ If $\{f_1, \dots, f_k\}$ is a basis of $L(E)$ then

$$G = \begin{pmatrix} f_1(P_1) & \dots & f_1(P_n) \\ \vdots & \ddots & \vdots \\ f_k(P_1) & \dots & f_k(P_n) \end{pmatrix} \in \mathbb{F}_q^{k \times n}$$

is a **generator** matrix of the code $\mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)$

Theorem I [Parameters of an AG code]

Let $\mathcal{C} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)$. If $\deg(E) < n$ then

$$k(\mathcal{C}) \geq \deg(E) + 1 - g \quad \text{and} \quad d(\mathcal{C}) \geq n - \deg(E)$$

Moreover, if $n > \deg(E) > 2g - 2$ then $k(\mathcal{C}) = \deg(E) - g + 1$.

Algebraic Geometry codes VII

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Dual of an AG code

Let:

- ω be a **differential form** with a simple pole and residue 1 at P_j for all $j = 1, \ldots, n$.
- K be the **canonical divisor** of ω.

Then

$$\mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)^\perp = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E^\perp)$$

with $\quad E^\perp = D_\mathcal{P} - E + K \quad$ and $\quad \deg(E^\perp) = n - \deg(E) + 2g - 2$

Theorem II [Parameters of the dual of an AG code]

Let $\mathcal{C} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)$. If $\deg(E) > 2g - 2$ then

$$k(\mathcal{C}^\perp) \geq n - \deg(E) - 1 + g \quad \text{and} \quad d(\mathcal{C}^\perp) \geq \deg(E) - 2g + 2$$

Moreover, if $n > \deg(E) > 2g - 2$ then $k(\mathcal{C}^\perp) = n - \deg(E) - 1 + g$

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t-ECP for AG codes I

➜ Consider the AG code

$$\mathcal{C} = \mathcal{C}_L\left(\mathcal{X}, \mathcal{P}, E\right)^{\perp}$$

Theorem [Pellikaan - 1992]

The pair of codes (A, B) defined by

$$A = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, F) \quad \text{and} \quad B = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E - F)$$

with $\deg(E) > \deg(F) \geq t + g$ is a t-ECP for $\mathcal{C}$.

➪ Such a pair **always exists** whenever

$$\deg(E) > 2g - 2 \quad \text{and } t = t^* = \left\lfloor \frac{d^* - 1 - g}{2} \right\rfloor .$$

where $d^* = \deg(E) - 2g + 2$ is the designed minimum distance of $\mathcal{C}$

t-ECP FOR AG CODES II

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COROLLARY [MAIN COROLLARY]

Let $\mathcal{C} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)^\perp$ and $B = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E - F)$ with $\deg(F) \geq t + g$.

And let us define $A_0 = (B * \mathcal{C})^\perp$. Then (A_0, B) is a t-ECP for $\mathcal{C}$

In order to compute a t-ECP for $\mathcal{C} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)$, it suffices to compute a code of type

$\mathcal{C}_L(\mathcal{X}, \mathcal{P}, E - F)$ for some divisor F with $\deg(F) \geq t + g$

Context of the Cryptosystem

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Public Key:

$$\mathcal{K}_{\text{pub}} = G \quad \text{and} \quad t^* = \left\lfloor \frac{d^* - g - 1}{2} \right\rfloor$$

where:

- G is a generator matrix of the **public code**:

$$\mathcal{C}_{\text{pub}} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)^{\perp}$$

- $d^* = \deg(E) - 2g + 2$ is the designed minimum distance of $\mathcal{C}_{\text{pub}}$

➜ Our t^* seems reasonable if $\mathcal{K}_{\text{secret}}$ is based on ECP.

$$t^* = \left\lfloor \frac{d^* - g - 1}{2} \right\rfloor \leq t = \left\lfloor \frac{d^* - 1}{2} \right\rfloor = \text{actual error-correction capability of } \mathcal{C}$$

➜ **Future work!!!**

The P-Filtration

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- Let $P = P_1$ be a point of the n-tuple $\mathcal{P}$.
- We focus on the sequence of codes:

$$\mathcal{B}_i := (\mathcal{C}_L(\mathcal{X}, \mathcal{P}, E - iP_1))_{i \in \mathbb{N}}$$

Which Elements of the Sequence do We know?

1. From a generator matrix of $\mathcal{C}_{\text{pub}} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)^\perp$ one can compute $\mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)$
   - Computed by **Gaussian elimination**.
2. $\mathcal{B}_0 = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)$.
3. $\mathcal{B}_1$ is the set of codewords of the code $\mathcal{B}_0$ which are zero at position P_1.
   - Computed by **Gaussian elimination**.

The codes $\mathcal{B}_0$ and $\mathcal{B}_1$ are **known**.

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How to compute $\mathcal{B}_2$?

- If $\frac{n}{2} > \deg(E)$, then $\mathcal{B}_1^{(2)} \subsetneq \mathbb{F}_q^n$.
- If $\deg(F - P) = \deg(E) - 1 \geq 2g + 1$, then

$$\mathcal{B}_1^{(2)} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E - P_1)^{(2)} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, 2E - 2P_1)$$

Thus, $\mathcal{B}_2$ is the solution space of the following problem

$$\mathbf{z} \in \mathcal{B}_1 \quad \text{and} \quad \mathbf{z} * \mathcal{B}_0 \subseteq (\mathcal{B}_1)^{(2)} \tag{1}$$

Proposition

Let $F,\ G$ be two divisors on $\mathcal{X}$ such that

$$\deg(F) \geq 2g \quad \text{and} \quad \deg(G) \geq 2g + 1$$

Then,

$$\mathcal{C}_L(\mathcal{X}, \mathcal{P}, F) * \mathcal{C}_L(\mathcal{X}, \mathcal{P}, G) = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, F + G)$$

Effective computation - Algorithm I

Construct $\mathcal{C}_L(\mathcal{X}, \mathcal{P}, E - F)$ with $\deg(F) \geq t^* + g$ from $\mathcal{C} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)^\perp$

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Theorem I: If we know $\mathcal{B}_{s-1}$ and $\mathcal{B}_s$ we can compute $\mathcal{B}_{s+1}$

$\mathcal{B}_{s+1}$ is the solution space of the following problem

$$\mathbf{z} \in \mathcal{B}_s \quad \text{and} \quad \mathbf{z} * \mathcal{B}_{s-1} \subseteq (\mathcal{B}_s)^{(2)} \tag{2}$$

If $s \geq 1$ and $\frac{n}{2} > \deg(E) \geq 2g + s + 1$.

$(t^* + g)$ repeated applications of **Theorem I** determines the code $\mathcal{B}_{t^*+g}$.

Effective computation - Algorithm II

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We can do **better** by **decreasing** the number of iterations and **relaxing** the parameters conditions $\Rightarrow$ Algorithm II

➜ **Algorithm I:**

$$\mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \mathcal{B}_3 \supseteq \ldots \supseteq \mathcal{B}_{t^*+g-1} \supseteq \mathcal{B}_{t^*+g}$$

Solve $(t^* + g)$ systems of linear equations

➜ **Algorithm II:**

$$\mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \mathcal{B}_4 \supseteq \ldots \supseteq \mathcal{B}_{\frac{t^*+g}{2}} \supseteq \mathcal{B}_{t^*+g}$$

Solve $2\lceil \log_2(t^* + g) \rceil + 2$ systems of linear equations

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ALGORITHM I VS. ALGORITHM II

CONSTRUCT $\mathcal{C}_L(\mathcal{X}, \mathcal{P}, E - F)$ WITH $\deg(F) \geq t^* + g$ FROM $\mathcal{C} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)^\perp$

➜ **Algorithm I:**

$$\mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \mathcal{B}_3 \supseteq \dots \supseteq \mathcal{B}_{t^*+g-1} \supseteq \mathcal{B}_{t^*+g}$$

Solve $(t^* + g)$ systems of linear equations

THEOREM I: IF WE KNOW $\mathcal{B}_{s-1}$ AND $\mathcal{B}_s$ WE CAN COMPUTE $\mathcal{B}_{s+1}$

$\mathcal{B}_{s+1}$ is the solution space of the following problem

$$\mathbf{z} \in \mathcal{B}_s \quad \text{and} \quad \mathbf{z} * \mathcal{B}_{s-1} \subseteq (\mathcal{B}_s)^{(2)}$$

If $s \geq 1$ and $\dfrac{n}{2} > \deg(E) \geq 2g + s + 1$.

ALGORITHM I VS. ALGORITHM II

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➜ **Algorithm II:**

$$\mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \mathcal{B}_4 \supseteq \dots \supseteq \mathcal{B}_{\frac{t^*+g}{2}} \supseteq \mathcal{B}_{t^*+g}$$

Solve $2\lceil \log_2(t^* + g)\rceil + 2$ systems of linear equations

THEOREM I: IF WE KNOW $\mathcal{B}_{\lfloor \frac{s}{2} \rfloor}$ AND $\mathcal{B}_{\lfloor \frac{s+1}{2} \rfloor}$ WE CAN COMPUTE $\mathcal{B}_s$

$\mathcal{B}_s$ is the solution space of the following problem

$$\mathbf{z} \in \mathcal{B}_s \quad \text{and} \quad \mathbf{z} * \mathcal{B}_0 \subseteq \mathcal{B}_{\lfloor \frac{s}{2} \rfloor} * \mathcal{B}_{\lfloor \frac{s+1}{2} \rfloor}$$

If $s \geq 1$ and $\frac{n}{2} > \deg(E) \geq 2g + s + 1$.

The Attack

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Public Key: $\mathcal{K}_{\text{pub}} = \mathcal{C}_{\text{pub}} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)^{\perp}$ and $t = \left\lfloor \dfrac{d^* - g - 1}{2} \right\rfloor$

The Algorithm: Suppose that $\dfrac{n}{2} \geq \deg(E)$.

Step 1. Determine the values g and $\deg(E)$ using the following Proposition.

Proposition

If $2g + 1 \leq \deg(E) < \dfrac{1}{2}n$.

Then, $\deg(E) = k(\mathcal{C}^{(2)}) - k(\mathcal{C})$ and $g = k(\mathcal{C}^{(2)}) - 2k(\mathcal{C}) + 1$

Step 2. Compute the code $\mathcal{B}_{t^*+g} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E - (t^* + g)P_1)$, using one of the algorithms described in §5.1

Step 3. Deduce an ECP from B.

Corollary: Let B of type $\mathcal{C}_L(\mathcal{X}, \mathcal{P}, E - F)$ with $\deg(F) \geq t^* + g$

Let us define $A_0 = (B * \mathcal{C})^{\perp}$. Then (A_0, B) is a t-ECP for $\mathcal{C} = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)^{\perp}$.

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From degenerate to non degenerate I

Unfortunately the codes

$$\mathcal{B}_i = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E - iP_1)$$

are **degenerated** for $i > 0$.

From degenerate to non degenerate II

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Aim of this section

How to computer another code

$$\hat{\mathcal{B}}_i = \mathcal{C}_L(\mathcal{X}, \mathcal{P}, E - F')$$

with:

1. $F' = F + (h)$ for some $h \in \mathbb{F}_q(\mathcal{X})$
2. $\mathrm{supp}(F') \cap \mathrm{supp}(D_{\mathcal{P}}) = \emptyset$

Remark: We do not need to compute h but just **prove its existence**.

➜ Them following result allows to compute a generator matrix of

$\hat{\mathcal{B}}_{t^*+g}$ from the codes $\mathcal{B}_{t^*+g}$ and $\mathcal{B}_{t^*+g+1}$.

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From degenerate to non degenerate III

Theorem

Let G be a generator matrix of $\mathcal{B}_{t^*+g}$ of the form:

$$G = \left(\begin{array}{c|c} 0 & \mathbf{c}_1 \\ \hline \mathbf{0} & G_1 \end{array}\right), \text{ where}$$

$$\begin{cases} \left(0 \mid \mathbf{c}_1 \right) \in \mathcal{B}_{t^*+g} \setminus \mathcal{B}_{t^*+g+1} \\ \left(\mathbf{0} \mid G_1 \right) = \text{gen. matrix of } \mathcal{B}_{t^*+g+1} \end{cases}$$

Then the following matrix is a generator matrix for $\hat{\mathcal{B}}_{t^*+g}$

$$\hat{G} = \left(\begin{array}{c|c} 1 & \mathbf{c}_1 \\ \hline \mathbf{0} & G_1 \end{array}\right)$$

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➜ The **costly part** of the attack is the computation of the code B

$\Longrightarrow$ We can apply one of the algorithms of §5.1

Computing:

1. a generator matrix of $\mathcal{C}^{(2)}$
2. and then apply Gaussian elimination to such matrix

costs

$$O\left(\binom{k}{2}n + \binom{k}{2}n^2\right) \sim O\left(k^2n^2\right) \text{ operations in } \mathbb{F}_q.$$

➜ Roughly speaking the cost of our attack is about $O\left((\lambda+1)n^4\right)$ where:

1. $\lambda =$ Linear systems to solve depending on the chosen algorithm from §5.1
2. The term $(\lambda + 1)$ is the cost of computing a non-degenerated code.

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EXAMPLES

- We summarize in the following tables the average running times of our algorithm for several codes.
- The attack has been implemented with MAGMA.
- The work factor **w** of and ISD attack is given. These work factors have been computed thanks to Christiane Peter's Software

Remark: **ISD's** average complexity is

$$O\left(k^2 n \frac{\binom{n}{t}}{\binom{n-k}{t}}\right) \text{ operations in } \mathbb{F}_q$$

Example I : Hermitian curves

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Hermitian Curve

The **Hermitian curve** $\mathcal{H}_r$ over $\mathbb{F}_q$ with $q = r^2$ is defined by the affine equation

$$Y^r + Y = X^{r+1}$$

➜ This curve has $P_\infty = (0:1:0)$ as the only point at infinity.

Take:

- ➜ $E = mP_\infty$
- ➜ $\mathcal{P}$ be the $n = q\sqrt{q} = r^3$ affine $\mathbb{F}_q$-rational points of the curve.

The following table considers different codes of type

$$\mathcal{C}_L(\mathcal{H}_r, \mathcal{P}, E)^\perp \text{ with } n > \deg(E) > 2g - 2.$$

q	g	n	k	t	**w**	key size	time
7^2	21	343	193	54	2^{84}	163 ko	74 s
9^2	36	729	404	126	2^{182}	833 ko	21 min
11^2	55	1331	885	168	2^{311}	2730 ko	67 min

Table: Comparison with Hermitian codes

w computed with Christiane Peters software

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Example II: Suzuki curves

Suzuki curves

The **Suzuki curves** are curves $\mathcal{X}$ defined over $\mathbb{F}_q$ by the following equation

$$Y^q - Y = X^{q_0}(X^q - X) \text{ with } q = 2q_0^2 \geq 8 \text{ and } q_0 = 2^r$$

This curve has exactly:

- $q^2 + 1$ rational places
- A single place at infinity P_∞.

Take:

- $E = mP_\infty$
- $\mathcal{P}$ be the q^2 rational points of the curve.

The following table considers several codes of type

$$\mathcal{C}_L(\mathcal{X}, \mathcal{P}, E)^\perp \text{ with } n > \deg(E) > 2g - 2.$$

q	g	n	k	t	**w**	key size	time
2^5	124	1024	647	64	2^{110}	1220 ko	30 min

Table: Comparison with Suzuki codes

w computed with Christiane Peters software

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- We constructed a **polynomial-time** algorithm which breaks the **McEliece scheme based on AG codes** whenever

$$2 < t \leq \left\lfloor \frac{d^* - g - 1}{2} \right\rfloor$$

- **COMPLEXITY:** $O\left(n^4\right)$

- **Future work:** using the concept of Error-Correcting Arrays (ECA) or well-behaving sequence obtain an attack for

$$t = \left\lfloor \frac{d^* - 1}{2} \right\rfloor$$

Thank you for your attention!

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