Variations on the Knapsack Generator

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ENS-PSL

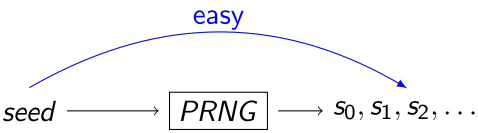
March 11th

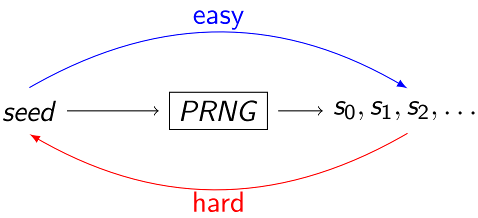
ENS

PRNG

$$seed \longrightarrow \boxed{PRNG} \longrightarrow s_0, s_1, s_2, \ldots$$

PRNG




easy
seed
PRNG
s0, s1, s2, . . .
hard

1. Definition of the Knapsack Generator
2. Attacks on the Knapsack Generator
3. Generalized Knapsack Generator

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1 Definition of the Knapsack Generator

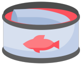
2 Attacks on the Knapsack Generator

3 Generalized Knapsack Generator

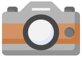
Knapsack Problem

Optimization Problem

$\leq C$



$\omega_1, \; p_1$



$\omega_2, \; p_2$

$\omega_3, \; p_3$

$\omega_4, \; p_4$

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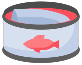
Knapsack Problem

Optimization Problem

$\leq C$



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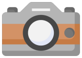

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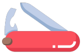

$\omega_3, \; p_3$

$\omega_4, \; p_4$

Goal: Finding bits u_i

$$\sum_{i=1}^{4} u_i \omega_i \leq C \text{ and } \sum_{i=1}^{4} u_i p_i \text{ maximal}$$

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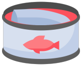
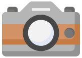
Subset Sum Problem (SSP)

Guessing Problem

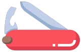
$= C$



ω_1



ω_2



ω_3

ω_4

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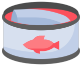

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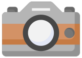
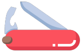


ω_2



ω_3

ω_4

Goal: Finding bits u_i

$$\sum_{i=1}^{4} u_i \omega_i = C$$

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Parameters:

- an integer n
- a vector of weights $\boldsymbol{\omega} = (\omega_0, \ldots, \omega_{n-1})$
- a target C
- a modulo M

The goal is finding $\mathbf{u}$ such that

$$\langle \mathbf{u}, \boldsymbol{\omega} \rangle = C \bmod M$$

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The closer M is to 2^n, the harder the problem is. For now $M = 2^n$

Knapsack Generator by Rueppel and Massey[1]

$$seed \longrightarrow \boxed{PRNG} \longrightarrow s_0, s_1, s_2, \ldots$$

Knapsack Generator by Rueppel and Massey[1]

$$\mathbf{u} \longrightarrow \boxed{\langle ., \boldsymbol{\omega} \rangle \text{ mod } M} \longrightarrow s_0, s_1, s_2, \ldots$$

Knapsack Generator by Rueppel and Massey[1]

$$\mathbf{u} \longrightarrow \boxed{\langle ., \omega \rangle \bmod M} \longrightarrow s_0, \cancel{s_1, s_2, \ldots}$$

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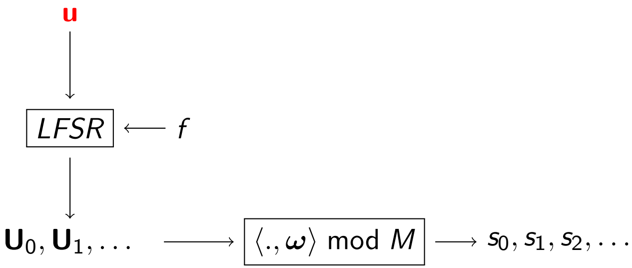

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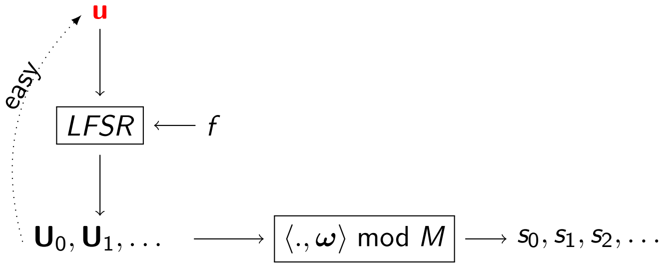

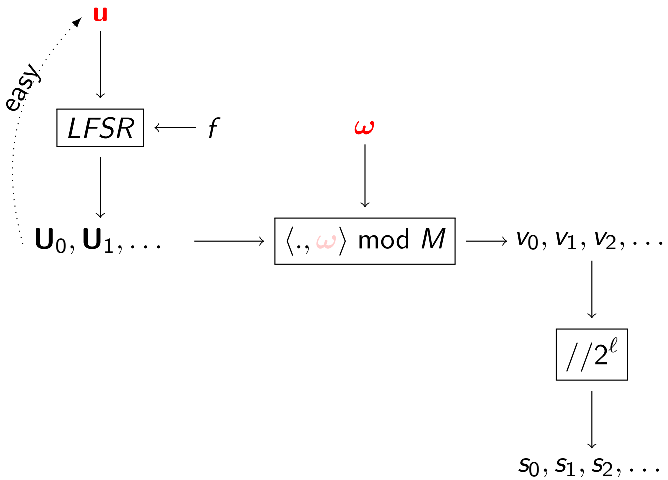

[1]Rueppel, R.A., Massey, J.L.: Knapsack as a nonlinear function. In: IEEE Intern. Symp. of Inform. Theory, vol. 46 (1985)

Formalization of the Knapsack Generator

Public	Secret
n and $\ell \in \mathbb{N}$	$\mathbf{u} \in \{0,1\}^n$
$f \in \mathbb{F}_2[X_1, \ldots, X_n]$	$\boldsymbol{\omega} \in \{0, \ldots, 2^n - 1\}^n$

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m is the number of outputs

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m is the number of outputs

Intermediate states	
$(u_i)_{i \geq n}$	$u_{n+i} = f(u_i, \ldots, u_{n+i-1})$
$(\mathbf{U}_i)_{0,\ldots,m-1}$	$\mathbf{U}_i = (u_i, \ldots u_{n+i-1})$

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$\mathbf{v} = (v_0,\ldots,v_{m-1})$	$v_i = \langle \mathbf{U}_i, \omega\rangle \bmod M$

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$\mathbf{v} = (v_0, \ldots, v_{m-1})$	$v_i = \langle \mathbf{U}_i, \omega \rangle \bmod M$
$\mathbf{s} = (s_0, \ldots, s_{m-1})$	$s_i = v_i // 2^\ell$
$\boldsymbol{\delta} = (\delta_0, \ldots, \delta_{m-1})$	$v_i = 2^\ell s_i + \delta_i$, $\lvert \boldsymbol{\delta} \rvert_\infty \leq 2^\ell$

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3. Generalized Knapsack Generator

The main flaw

$= $  $(\mathbf{u})$ $+$  (ω)

$n(1+n)$ bits = n bits + n^2 bits

The main flaw

$n(1+n)$ bits

$=$

$(\mathbf{u})$

n bits

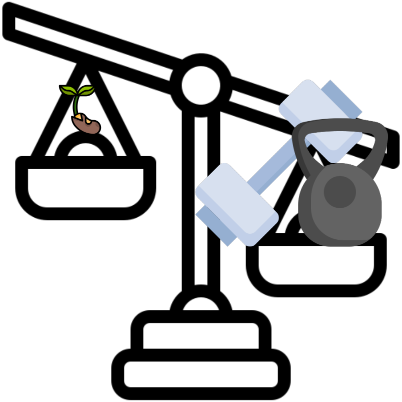
$\uparrow$

SMALL

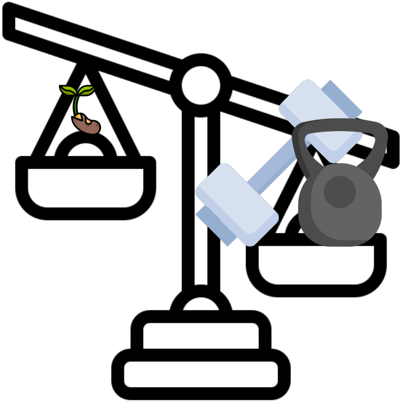
$+$

(ω)

n^2 bits



The secret is unbalanced.



The secret is unbalanced.
For a secret of ~ 1024 bits, the seed $(\mathbf{u})$ is only made of 32 bits.

Layout

ApproxWeights($\mathbf{u}, \mathbf{s}(\mathit{short})$):

???

Return($\boldsymbol{\omega}'$)

Check Consistency ($\mathbf{u}', \boldsymbol{\omega}', \mathbf{s}(\mathit{long})$):

$\mathbf{s}' = \mathit{PRNG}(\mathbf{u}', \boldsymbol{\omega}')$

Return Boolean($\mathbf{s}'$ is close to $\mathbf{s}$)

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Full Attack($\mathbf{s}$):

For $\mathbf{u}' \in \{0,1\}^n$:

$\boldsymbol{\omega}' =$ ApproxWeights($\mathbf{u}', \mathbf{s}(\textit{short})$)

If Check Consistency($\mathbf{u}', \boldsymbol{\omega}', \mathbf{s}(\textit{long})$) $=$ True

Return ($\mathbf{u}', \boldsymbol{\omega}'$)

End If

End For

- If $\mathbf{v} = (v_0, \ldots v_{n-1})$, $\|\mathbf{v}\|_\infty = \max_{i \in \{0,\ldots,n-1\}} |v_i|$
- If M is a matrix, $\|M\|_\infty = \max_{\|\mathbf{v}\|_\infty = 1} \|\mathbf{v}M\|_\infty$

Hence

$$\|\mathbf{v}M\|_\infty \leq \|\mathbf{v}\|_\infty \|M\|_\infty$$

Attack of Knellwolf and Meier [2]

$$U = \begin{pmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \\ \dots \\ \mathbf{U}_{m-1} \end{pmatrix}$$

[2] Knellwolf, S., & Meier, W. (2011). Cryptanalysis of the knapsack generator. FSE 2011

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$$\begin{aligned} \omega &= \mathbf{v} T \bmod M \\ &= 2^\ell \mathbf{s} T + \boldsymbol{\delta} T \bmod M \end{aligned}$$

$$\omega - 2^\ell \mathbf{s} T = \boldsymbol{\delta} T \bmod M$$

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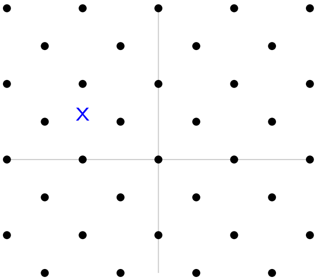
$$\begin{aligned} \boldsymbol{\omega} &= \mathbf{v}T \bmod M \\ &= 2^{\ell}\mathbf{s}T + \boldsymbol{\delta}T \bmod M \end{aligned}$$

$$\boldsymbol{\omega} - 2^{\ell}\mathbf{s}T = \boldsymbol{\delta}T \bmod M$$

Goal : Construct small $\hat{T}$ such that $\|\boldsymbol{\delta}\hat{T}\|_{\infty} < M$

[2]Knellwolf, S., & Meier, W. (2011). Cryptanalysis of the knapsack generator. FSE 2011

Lattice Interlude: CVP and Babai Rounding



$\mathsf{x} = (-2, 1.1)$

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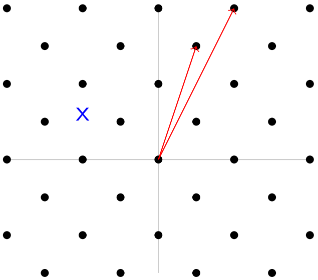
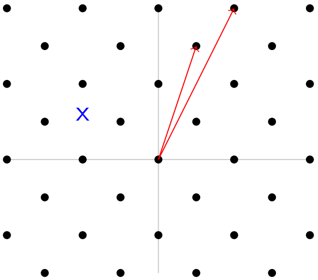


$x = (-2, 1.1)$

$$M = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \text{ and}$$

$$\mathcal{L} = \{\alpha M \mid \alpha \in \mathbb{Z}^2\}$$

Lattice Interlude: CVP and Babai Rounding



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β such that $\mathsf{x} = \beta M$, $\beta = (5.1, -3.55)$

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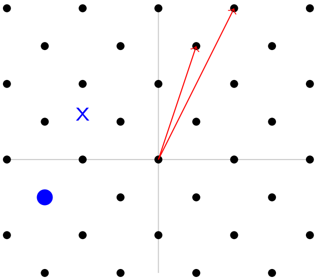

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$x' = \lfloor \beta \rceil M = (-3, -1)$

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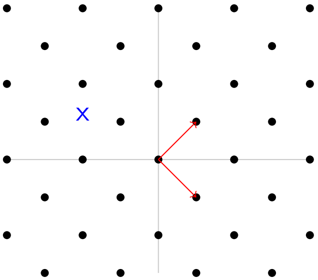

$\mathsf{x} = (-2, 1.1)$

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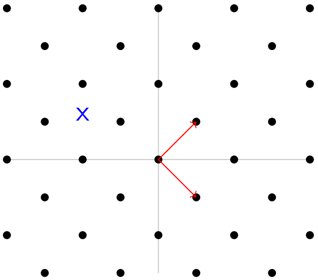

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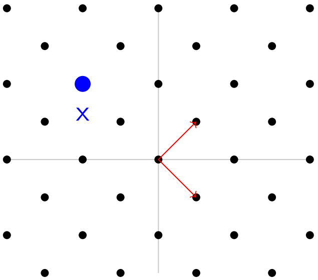

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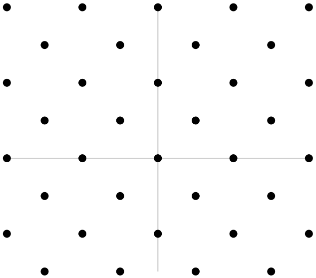
$x' = \lfloor \beta \rceil M = (-2, 2)$

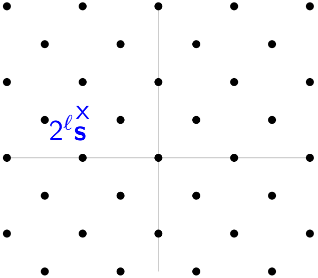
I have $\mathbf{v} = \boldsymbol{\omega} U \bmod M$



$\mathcal{L} = \{\alpha U \bmod M \mid \alpha \in \mathbb{Z}^n\}$

My variation

I have $\mathbf{v} = \boldsymbol{\omega} U \bmod M$ and $\mathbf{v} = 2^\ell \mathbf{s} + \boldsymbol{\delta}$ with $\boldsymbol{\delta}$ small



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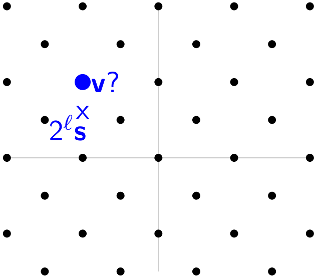

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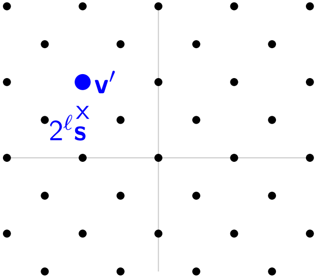

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Failed, this is not $\mathbf{v}$, we call it $\mathbf{v}'$

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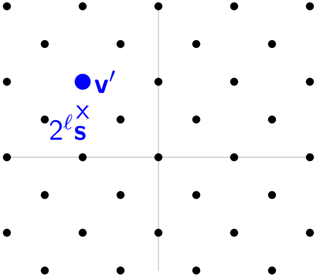

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Failed, this is not $\mathbf{v}$, we call it $\mathbf{v}'$

We compute $\boldsymbol{\omega}$' as

$$\boldsymbol{\omega}' U = \mathbf{v}' \bmod M$$

Why is $\boldsymbol{\omega}'$ close to $\boldsymbol{\omega}$?

Why does it work ? First Explanation

$(\omega - \omega')U = \mathbf{v} - \mathbf{v}' \bmod M$

Why does it work ? First Explanation

$$(\omega - \omega')U = \mathbf{v} - \mathbf{v}' \bmod M \quad \Leftrightarrow (\omega - \omega') = (\mathbf{v} - \mathbf{v}')\hat{T} \bmod M$$

Why does it work ? First Explanation

$$\begin{aligned}(\omega - \omega')U = \mathbf{v} - \mathbf{v}' \bmod M \quad &\Leftrightarrow (\omega - \omega') = (\mathbf{v} - \mathbf{v}')\hat{T} \bmod M \\ &\Rightarrow \|\omega - \omega'\|_\infty \leq \|\hat{T}\|_\infty \|\mathbf{v} - \mathbf{v}'\|_\infty\end{aligned}$$

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\end{aligned}
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In KW case: $\|\omega - 2^\ell \mathbf{s}\hat{T}\|_\infty \simeq \|\hat{T}\|_\infty \|\boldsymbol{\delta}\|_\infty$

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\end{aligned}
$$

In KW case: $\|\omega - 2^\ell \mathbf{s}\hat{T}\|_\infty \simeq \|\hat{T}\|_\infty \|\boldsymbol{\delta}\|_\infty$
But in our case $\|\omega - \omega'\|_\infty \ll \|\hat{T}\|_\infty \|\mathbf{v} - \mathbf{v}'\|_\infty$, precisely
$\|\omega - \omega'\|_\infty \leq \|\mathbf{v} - \mathbf{v}'\|_\infty$

Why does it work ? Second Explanation

I already have $\|\mathbf{v} - \mathbf{v}'\|_\infty \leq 2^{\ell+1} \Leftarrow \|\boldsymbol{\omega} - \boldsymbol{\omega}'\|_\infty \leq \frac{2^{\ell+1}}{\|U\|_\infty}$ (1)

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If I call $\mathcal{L} = \{\alpha U \bmod M \mid \alpha \in \mathbb{Z}^n\}$, then

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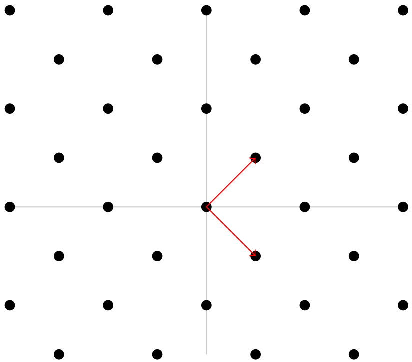
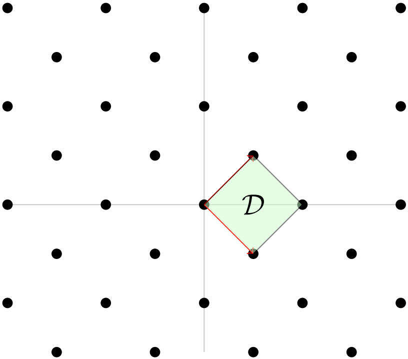
$$|\mathcal{B}| = (2\lfloor \frac{2^{\ell+1}}{\|U\|_\infty} \rfloor - 1)^n$$



Lattice Interlude n2: Fundamental domain



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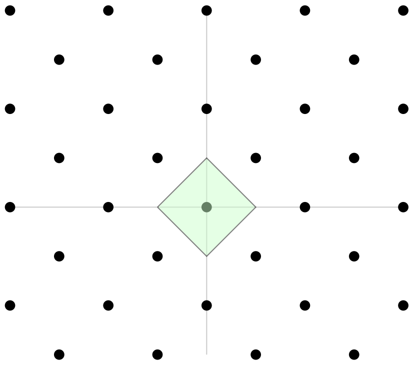

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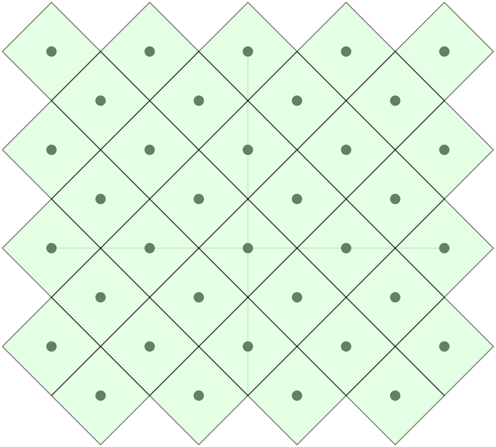

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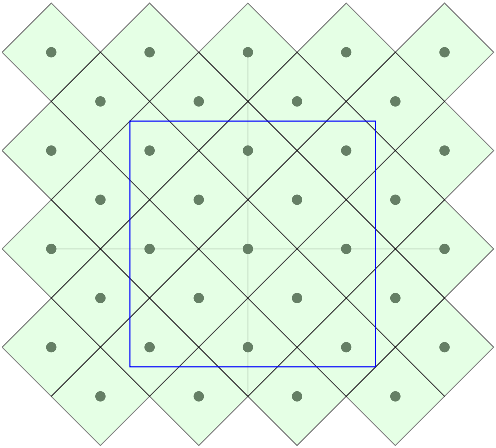

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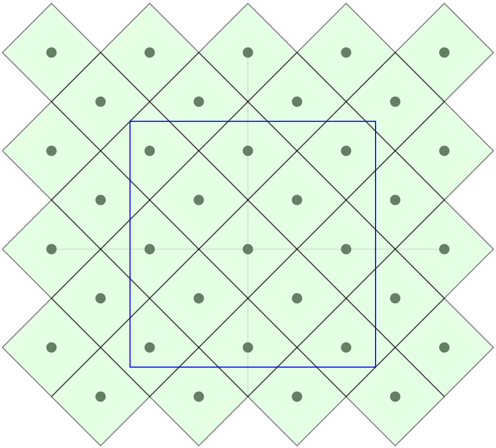

$$\frac{vol(\textcolor{blue}{rectangle})}{vol(\mathcal{D})} = 12.5 \sim 13$$

$$|\mathcal{B}| = (2\lfloor \frac{2^{\ell+1}}{\|U\|_\infty} \rfloor - 1)^n$$

$$|\mathcal{A}| \simeq \frac{2^n(2^{\ell+1} - 1)^n}{2^{n-m}}$$

For $n = 32$ and $m = 40$ we obtain $|\mathcal{B}| \geq |\mathcal{A}|$ for $\ell \leq 14$.

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ℓ	5	10	15	20	25
$log_2(\|\omega - 2^\ell \hat{T}\|_\infty)$	9.9	14.9	19.8	24.7	~~31~~
$log_2(\|\omega - \omega'\|_\infty)$	3.6	8.7	13.6	18.7	~~31~~

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2. Attacks on the Knapsack Generator
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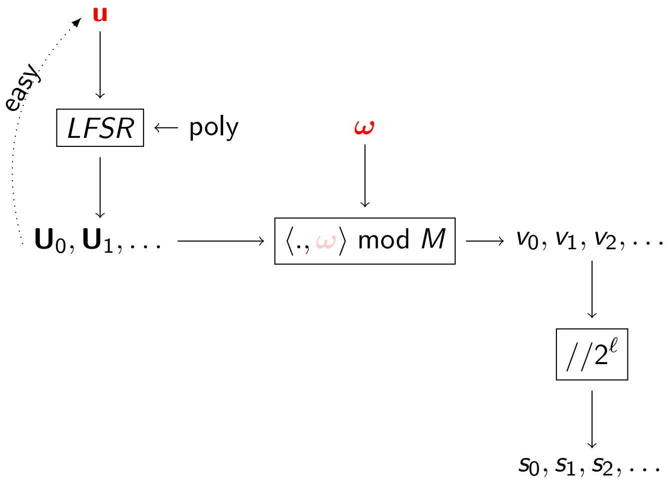
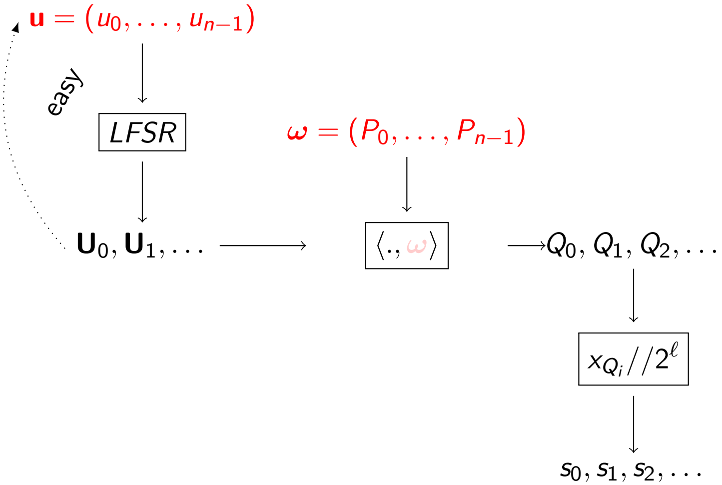
Knapsack Generator by Rueppel and Massey



Generalized Knapsack Generator by Von zur Gathen and Shparlinski[3]



[3] von zur Gathen, J., & Shparlinski, I. E. . Predicting subset sum pseudorandom generators. In Selected Areas in Cryptography: 11th International Workshop, SAC 2004.

Formalization of the Generalized Knapsack Generator

Public	Secret
n and $\ell \in \mathbb{N}$	$\mathbf{u} = (u_0, \ldots, u_{n-1}) \in \{0,1\}^n$
$f \in \mathbb{F}_2[X_1, \ldots, X_n]$	$\boldsymbol{\omega} = (P_0, \ldots, P_{n-1}) \in \mathcal{E}^n$
$\mathcal{E}$ elliptic curve over $\mathbb{F}_p$	

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n and $\ell \in \mathbb{N}$ $f \in \mathbb{F}_2[X_1, \ldots, X_n]$ $\mathcal{E}$ elliptic curve over $\mathbb{F}_p$	$\mathbf{u} = (u_0, \ldots, u_{n-1}) \in \{0,1\}^n$ $\boldsymbol{\omega} = (P_0, \ldots, P_{n-1}) \in \mathcal{E}^n$

m is the number of outputs

Intermediate states	
$(u_i)_{i \geq n}$	$u_{n+i} = f(u_i, \ldots, u_{n+i-1})$
Q_j	$Q_j = \sum_{i=0}^{n-1} u_{i+j} P_i$
s_i	$s_i = x_{Q_i} // 2^\ell$
δ_i	$x_{Q_i} = 2^\ell s_i + \delta_i$, $\delta_i \leq 2^\ell$

$$(x, y) \text{ such that } y^2 = x^3 + ax + b \bmod p$$

For x_0:

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- there is no P such that $x_P = x_0$

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For $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$

- $s = \frac{y_P - y_Q}{x_P - x_Q}$
- $x_R = s^2 - x_P - x_Q$
- $y_R = y_P - s(x_P - x_R)$
- $P + Q = -R$

$\mathbb{Z}$	$\mathcal{E}$
$\|x - x'\| < 2^\ell$ $\|y - y'\| < 2^\ell$	$\|x_P - x_{P'}\| < 2^\ell$ $\|x_Q - x_{Q'}\| < 2^\ell$

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Elliptic curve over $\mathbb{F}_p$ Interlude - part 2

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$\lvert x - x' \rvert < 2^{\ell}$ $\lvert y - y' \rvert < 2^{\ell}$	$\lvert x_P - x_{P'} \rvert < 2^{\ell}$ $\lvert x_Q - x_{Q'} \rvert < 2^{\ell}$
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Elliptic curve over $\mathbb{F}_p$ Interlude - part 2

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$\lvert x - x' \rvert < 2^\ell$ $\lvert y - y' \rvert < 2^\ell$	$\lvert x_P - x_{P'} \rvert < 2^\ell$ $\lvert x_Q - x_{Q'} \rvert < 2^\ell$
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- If $P' + Q' = \pm(P+Q)$, $\mathbb{P}_{P',Q'}(\lvert x_{P+Q} - x_{P'+Q'} \rvert < 2^\ell) = 1$
- If $P' + Q' \neq \pm(P+Q)$, $\mathbb{P}_{P',Q'}(\lvert x_{P+Q} - x_{P'+Q'} \rvert < 2^\ell)$
  $= \mathbb{P}_R(\lvert x_{P+Q} - x_R \rvert < 2^\ell) = \frac{2^\ell}{\lvert \mathcal{E} \rvert}$

The problem

$$\begin{pmatrix} P_0 & P_1 & \dots & P_{n-1} \end{pmatrix} \times \begin{pmatrix} u_0 & u_1 & \dots & u_{n-1} \\ u_1 & u_2 & \dots & u_n \\ & & \ddots & \\ u_{n-1} & u_n & \dots & u_{2n-2} \end{pmatrix} = \begin{pmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{n-1} \end{pmatrix}$$

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Naive attack : Guess $\mathbf{u}$ and $\boldsymbol{\delta}$: $\mathcal{O}(2^n \times 2^{n\ell})$ operations

The problem

$$\begin{pmatrix} P_0 & P_1 & \dots & P_{n-1} \end{pmatrix} \times \begin{pmatrix} u_{i_1} & u_{i_1+1} & \dots & u_{i_1+n-1} \\ u_{i_2} & u_{i_2+1} & \dots & u_{i_2+n-1} \\ & & \ddots & \\ u_{i_n} & u_{i_n+1} & \dots & u_{i_n+n-1} \end{pmatrix} = \begin{pmatrix} Q_{i_1} \\ Q_{i_2} \\ \vdots \\ Q_{i_n} \end{pmatrix}$$

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I want to go here in less than: $\mathcal{O}(2^n \times 2^{n\ell})$ operations

Two steps:

- Finding $n/2$ "good triplets" i, j, k such that $\mathbf{U}_i + \mathbf{U}_j = \mathbf{U}_k$ (in $\mathbb{Z}$!)
- For each triplet, retrieving Q_i, Q_j by bruteforce.

Bruteforcing Q_i, Q_j

I have 3 points Q_i, Q_j, Q_k that I do not know but I know:

- s_i, s_j, s_k the leading bits of xQ_i, xQ_j, xQ_k
- the relation $Q_i + Q_j = Q_k$

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$A_i = \{R_i \mid x_{R_i}//2^\ell = s_i\}$ and $A_j = \{R_j \mid x_{R_j}//2^\ell = s_j\}$

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$$\mathbb{P}(\exists (R_i, R_j) \in A_1 \times A_2 \mid x_{R_i+R_j}//2^\ell = s_k \wedge (R_i, R_j) \neq \pm(Q_i, Q_j))$$
$$\simeq |A_i \times A_j| \times \frac{2^\ell}{|\mathcal{E}|} \simeq \frac{2^{3\ell}}{|\mathcal{E}|}$$

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$$\simeq |A_i \times A_j| \times \frac{2^\ell}{|\mathcal{E}|} \simeq \frac{2^{3\ell}}{|\mathcal{E}|}$$

If ℓ small enough, I can bruteforce (Q_i, Q_j) and $(-Q_i, -Q_j)$ out of $A_i \times A_j$ in $\mathcal{O}(2^{2\ell})$ operations using s_k as a filter. They are not distinguishable.

$(\mathbf{U}_i, \mathbf{U}_j, \mathbf{U}_k) \in \{0,1\}^n$, if $n = 1$:

$\mathbf{U}_i$	0	0	0	0	1	1	1	1
$\mathbf{U}_j$	0	0	1	1	0	0	1	1
$\mathbf{U}_k$	0	1	0	1	0	1	0	1
$\mathbf{U}_i + \mathbf{U}_j$	0	0	1	1	1	1	2	2

$$\mathbb{P}(\mathbf{U}_i + \mathbf{U}_j = \mathbf{U}_k) = 3/8$$

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If $n \geq 1$, $\mathbb{P}(\mathbf{U}_i + \mathbf{U}_j = \mathbf{U}_k) = (3/8)^n$

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$\mathbf{U}_i \in A$, $\mathbf{U}_j \in B$, $\mathbf{U}_k \in C$, $|A| = |B| = |C| = N$

$$\mathbb{E}(\text{ good triplets }) = N^3 \left(\frac{3}{8}\right)^n$$

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For now $N \simeq \left(\frac{8}{3}\right)^{n/3}$

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Then we often have $w(\mathbf{U}_i) \simeq w(\mathbf{U}_j) \simeq n/3$ and $w(\mathbf{U}_k) \simeq 2n/3$

A Sub-Quadratic Algorithm

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Then we often have $w(\mathbf{U}_i) \simeq w(\mathbf{U}_j) \simeq n/3$ and $w(\mathbf{U}_k) \simeq 2n/3$

```
function FindTriplet(A, B, C, ε)
    A′ ← {U_i ∈ A | w(U_i) ≤ n/3 + ε}
    B′ ← {U_j ∈ B | w(U_j) ≤ n/3 + ε}
    for all U_i, U_j ∈ A′ × B′ do
        if U_i + U_j ∈ C then
            return (U_i, U_j, U_k)
    return ⊥
```

For $\epsilon = 1/6$, the algorithm succeed with overwhelming probability in time $\mathcal{O}(N^{1.654}) \simeq \mathcal{O}(2^{0.78n})$.

For all $(u_0, \dots, u_{n-1})$ in $\{0,1\}^n$:

- derive all the $\mathbf{U}_i$ and find $n/2$ good triplets in $\mathcal{O}(2^{0.78n})$
- for each good triplet derive (Q_i, Q_j) and $(-Q_i, -Q_j)$ in $\mathcal{O}(2^{2\ell})$
- derive the P_i's for the $2^{n/2-1}$ possible signs combinations
- check consistency

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- derive the P_i's for the $2^{n/2-1}$ possible signs combinations
- check consistency

The complexity is

$$\mathcal{O}(2^n \times (2^{0.78n} + (n/2 \times 2^{2\ell}) + 2^{n/2-1}))$$

that is to say $\mathcal{O}(2^{1.78n})$ binary operations (with $\ell = \log_2(n)$).

When $n = 16$ and the initial sequence $(u_0, \ldots, u_{n-1})$ is known.

- When $|\mathcal{E}| = 65111$.

ℓ	1	2	3	4	5	6
m	1000	1000	1000	1000	1000	1885
time	$6.9s$	$5.3s$	$5.6s$	$5.02s$	$5.7s$	$26.7s$

- When $|\mathcal{E}| = 1099510687747$.

ℓ	1	2	3	4	5	6	7	8	9
m	1885	1885	1885	1885	1885	1885	1885	1885	1750
time	$2.1s$	$2.1s$	$2.08s$	$2.5s$	$2.6s$	$2.1s$	$3.5s$	$8.3s$	$26.7s$