A New Algebraic Approach to the Regular Syndrome Decoding Problem and Implications for PCG Constructions

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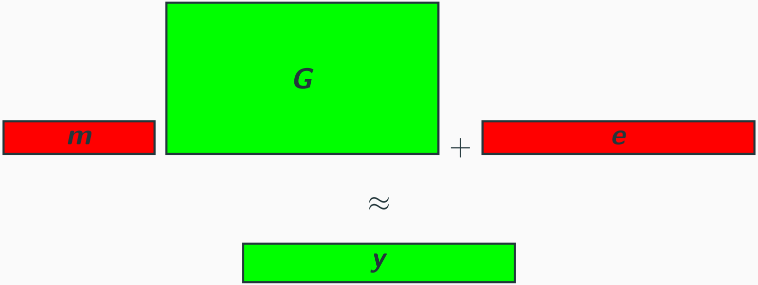
Decoding Problem over $\mathbb{F}_q$ (aka Primal LPN)

Given full-rank $\boldsymbol{G} \in \mathbb{F}_q^{k \times n}$, distinguish

- $\boldsymbol{y} = \boldsymbol{m}\boldsymbol{G} + \boldsymbol{e}$, $\boldsymbol{m} \in \mathbb{F}_q^k$, error $\boldsymbol{e} \sim \chi$
- $\boldsymbol{y} \sim \mathcal{U}(\mathbb{F}_q^n)$



Our setting

Bounded number of samples $n = k^{1+\alpha}$, $0 < \alpha < 1$

Error $\boldsymbol{e}$ of low Hamming weight, $|\boldsymbol{e}| = t$

$\rightarrow$ Coding theory point of view ! Length n, dim. k, code rate $R \stackrel{def}{=} k/n$

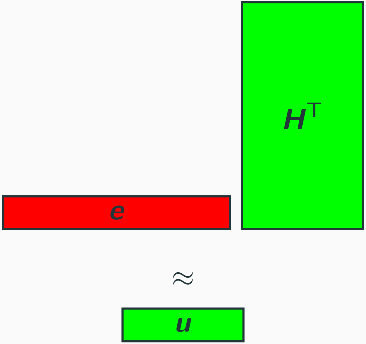
Underlying code $\mathcal{C}$

$$\mathcal{C} \stackrel{def}{=} \left\{\boldsymbol{mG},\ \boldsymbol{m} \in \mathbb{F}_q^k\right\} = \left\{\boldsymbol{x} \in \mathbb{F}_q^n,\ \boldsymbol{xH}^\mathsf{T} = \boldsymbol{0}\right\},\ \boldsymbol{H} \in \mathbb{F}_q^{(n-k)\times n}$$

Syndrome Decoding Problem (aka Dual LPN)

Given full-rank $\boldsymbol{H} \in \mathbb{F}_q^{(n-k)\times n}$, distinguish

- $\boldsymbol{u} = \boldsymbol{e}\boldsymbol{H}^\mathsf{T} \in \mathbb{F}_q^{n-k}, \ \boldsymbol{e} \sim \chi$
- $\boldsymbol{u} \sim \mathcal{U}(\mathbb{F}_q^{n-k})$



Some use cases

- Symmetric crypto [HB01]
- PKE: Alekhnovich scheme [Ale03]

Correlated randomness for secure MPC, ZK proofs...
Pseudorandom correlation generators (PCGs). Ex., for Vector OLE [Boy+19]:

1. Function Secret Sharing $\to$ Additive shares of sparse $\boldsymbol{e}$
2. Expansion with LPN PRG $(\boldsymbol{m}, \boldsymbol{e}) \mapsto \boldsymbol{mG} + \boldsymbol{e}$ or $\boldsymbol{e} \mapsto \boldsymbol{eH}^{\top}$

[HB01] Hopper and Blum. "Secure Human Identification Protocols". *Advances in Cryptology — ASIACRYPT 2001*.

[Ale03] Alekhnovich. "More on Average Case vs Approximation Complexity".

[Boy+19] Boyle et al. *Compressing Vector OLE*.

Parameters for PCGs

~~Code-based~~ crypto

LOW noise rate (inverse poly, not constant) $\rightarrow$ Very large sizes

Possibly large field (typically $\mathbb{F}_{2^{128}}$)

Rate R depends on PRG

- Very low for $(\boldsymbol{m}, \boldsymbol{e}) \mapsto \boldsymbol{mG}$ ("Primal")
- Constant for $\boldsymbol{e} \mapsto \boldsymbol{eH}^\top$ ("Dual")

Ex. "Primal", $\lambda = 128$: $[\mathbb{F}_2,\ n = 2^{22},\ k = 67440,\ t = 4788]$ [Boy+19]; [Liu+22]

[Liu+22] Liu et al. *The Hardness of LPN over Any Integer Ring and Field for PCG Applications*.

Regular Syndrome Decoding

Assume $n = N \times t$ for some $N \in \mathbb{N}$ (blocksize)

Regular distribution [AFS05]

- For $1 \leq i \leq t$, sample $\mathbf{e}_i \in \mathbb{F}_q^N$ **random of weight 1**
- Error is $\mathbf{e} \stackrel{def}{=} (\mathbf{e}_1, \ldots, \mathbf{e}_t) \in \mathbb{F}_q^n$

Introduction in Secure Computation [Haz+18]

Now in many PCG protocols [Boy+19]; [Wen+20]; [Yan+20]...

→ Reduce Function Secret Sharing cost

[AFS05] Augot, Finiasz, and Sendrier. "A Family of Fast Syndrome Based Cryptographic Hash Functions". *MYCRYPT 2005*.

[Haz+18] Hazay et al. *TinyKeys: A New Approach to Efficient Multi-Party Computation*.

Known attacks on RSD

Do NOT exploit regular distribution:

- "Folklore attack" and ISD algorithms [Pra62]; [MMT11]; [MO15]...
- Statistical Decoding [Jab01] (recently improved by [Car+22])

What about algebraic techniques ?

[Pra62] Prange. "The use of information sets in decoding cyclic codes".

[Jab01] Jabri. "A Statistical Decoding Algorithm for General Linear Block Codes".

[Car+22] Carrier et al. *Statistical Decoding 2.0: Reducing Decoding to LPN*.

Algebraic attacks

Generic technique in cryptanalysis:

- Model scheme or hard problem as polynomial system
- Solve it ! (Gröbner Bases, linearization)

This talk

Algebraic attack on RSD

- Good for low rates used in “Primal”
- Polynomial system + detailed analysis

(Naive) algebraic system

Modeling regular structure

Polynomial ring $R \overset{def}{=} \mathbb{F}_q[(e_{i,j})_{i,j}]$ in n variables, block $\boldsymbol{e}_i \overset{def}{=} (e_{i,1}, \ldots, e_{i,N}) \in \mathbb{F}_q^N$

Coordinates $\in \mathbb{F}_q$ (field equations)

$$\forall i,\ \forall j,\ e_{i,j}^q - e_{i,j} = 0. \tag{1}$$

One $\neq 0$ coordinate per block

$$\forall i,\ \forall j_1 \neq j_2,\ e_{i,j_1} e_{i,j_2} = 0. \tag{2}$$

Over $\mathbb{F}_2$, this coordinate is 1

$$\forall i,\ \sum_{j=1}^{N} e_{i,j} = 1. \tag{3}$$

We consider quadratic system $\mathcal{Q} \overset{def}{=} (1) \cup (2) \cup (3)$

Adding parity-check equations

Linear equations in the $e_{i,j}$'s from $\boldsymbol{eH}^\top = \boldsymbol{u}$:

Parity-checks

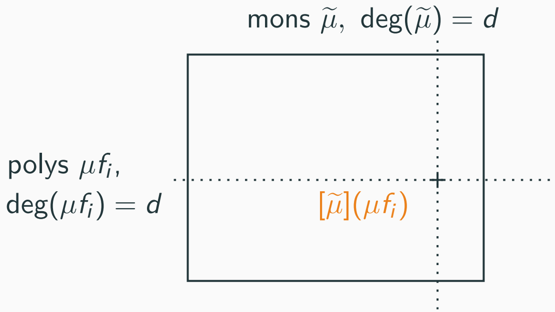
$$\mathcal{P} \stackrel{def}{=} \{\forall i \in \{1..n-k\},\ \langle \boldsymbol{e}, \boldsymbol{h}_i \rangle - u_i = 0\}\,.$$

Final system $\mathcal{S} \stackrel{def}{=} \mathcal{P} \cup \mathcal{Q}$

Set of solutions to $\mathcal{S}$ = Set of solutions to RSD (let's say 1)

1) Multiply by monomials 2) Linear Algebra up to some degree D

Macaulay matrix $\boldsymbol{M}_d$, $d \leq D$:



Cost $\exp(D)$

Need to estimate solving degree D

Analyzing $\mathcal{S}$

Approach

Recall that $\mathcal{S} = \underbrace{\{\text{parity-checks}\}}_{\mathcal{P}} \cup \underbrace{\{\text{regular structure}\}}_{\mathcal{Q}}$

$$\mathcal{P} = \{\forall i \in \{1..n-k\},\ \langle \boldsymbol{e}, \boldsymbol{h}_i \rangle - u_i\}$$

$$\mathcal{Q} = \{\forall i \in \{1..t\}, \forall j \in \{1..N\},\ e_{i,j}^2 - e_{i,j}\} \cup \{\forall i, \forall j_1 \neq j_2,\ e_{i,j_1} e_{i,j_2}\} \cup \{\forall i,\ \textstyle\sum_{j=1}^{N} e_{i,j} - 1\}$$

- To keep internal structure, treat $\mathcal{P}$ and $\mathcal{Q}$ separately
- Focus on homogeneous parts: $\langle \mathcal{S}^{(h)} \rangle = \langle \mathcal{P}^{(h)} \rangle + \langle \mathcal{Q}^{(h)} \rangle$

Solving degree from Hilbert series

Polynomial ring $R \overset{def}{=} \mathbb{F}_q[(e_{i,j})_{i,j}]$, $R = \oplus_{d\in\mathbb{N}} R_d$ hom. components

Hom. ideal $I \overset{def}{=} \langle f_1, \ldots, f_m \rangle$, $I_d \overset{def}{=} I \cap R_d$

Hilbert series (HS) of I

Contains properties of I we need

$\to$ Find Hilbert series for $\langle \mathcal{S}^{(h)} \rangle$ then deduce D

More formally

$$\mathcal{H}_{R/I}(z) \overset{def}{=} \sum_{d\in\mathbb{N}} \dim\left(R_d/I_d\right) z^d$$

0-dimensional ideal ($\mathcal{H}_{R/I}(z)$ is a polynomial): $H(I) \overset{def}{=} \min\{\delta \in \mathbb{N}, I_\delta = R_\delta\}$

Structural part $\mathcal{Q}$

Only depends on regular distribution. We analyze $q = 2$ (*e.g.* we can use (3))

$$\mathcal{Q}^{(h)} = \underbrace{\{\forall i \in \{1..t\}, \forall j \in \{1..N\},\ e_{i,j}^2\}}_{(1)} \cup \underbrace{\{\forall i, \forall j_1 \neq j_2,\ e_{i,j_1} e_{i,j_2}\}}_{(2)} \cup \underbrace{\{\forall i,\ \sum_{j=1}^{N} e_{i,j}\}}_{(3)}$$

HS 1

We have $\dim(R_d / \langle \mathcal{Q}^{(h)} \rangle_d) = \binom{t}{d}(N-1)^d.$ Thus,

$$\mathcal{H}_{R/\langle \mathcal{Q}^{(h)} \rangle}(z) = (1 + (N-1)z)^t$$

Proof (monomial counting).

Using (1) and (2), squarefree + at most one variable per $\boldsymbol{e}_i$ block

Using (3), we get rid of one variable per $\boldsymbol{e}_i$ block □

Random part $\mathcal{P}$

We have $\mathcal{P}^{(h)} = \{\boldsymbol{e}\boldsymbol{H}^\top\}$. By assumption on $\boldsymbol{H}$, "random" linear equations

- but we want "randomness" in $R/\langle \mathcal{Q}^{(h)} \rangle$
- here, randomness means (semi)-regularity:

Semi-regularity over $\mathbb{F}_2$ [Bar04]

Let $S \stackrel{def}{=} \mathbb{F}_2[\mathbf{e}]/\langle \mathbf{e}^2 \rangle$, $\mathcal{F} = \{f_1, \ldots, f_m\}$ homogeneous, 0-dim, index $d_{\langle \mathcal{F} \rangle}$
System $\mathcal{F}$ is semi-regular over $\mathbb{F}_2$ if $\langle \mathcal{F} \rangle \neq S$ and if

$$\forall i, \ \deg(g_i f_i) < d_{\langle \mathcal{F} \rangle}, \ g_i f_i = 0 \in S/\langle f_1, \ldots, f_{i-1} \rangle \Rightarrow g_i = 0 \in S/\langle f_1, \ldots, f_i \rangle \quad (4)$$

In this paper, we adapt it to $R/\langle \mathcal{Q}^{(h)} \rangle$ instead of $R/\langle \mathbf{e}^2 \rangle$

[Bar04] Bardet. "Étude des systèmes algébriques surdéterminés. Applications aux codes correcteurs et à la cryptographie".

Combining everything

Semi-regular HS are known ! (write exact sequences from (4))

Assumption

We assume semi-regularity of $\mathcal{P}^{(h)}$ with our new definition

We have $\langle \mathcal{S}^{(h)} \rangle = \langle \mathcal{P}^{(h)} \rangle + \langle \mathcal{Q}^{(h)} \rangle$, we know $\mathcal{H}_{R/\langle \mathcal{Q}^{(h)} \rangle}$. We want $\mathcal{H}_{R/\langle \mathcal{S}^{(h)} \rangle}$
Under Assumption, we get

$$\mathcal{H}_{R/\langle \mathcal{S}^{(h)} \rangle}(z) = \frac{\mathcal{H}_{R/\langle \mathcal{Q}^{(h)} \rangle}(z)}{(1+z)^{n-k}}$$

HS for $\mathcal{S}^{(h)}$ (under Assumption + using HS 1)

$$\mathcal{H}_{R/\langle \mathcal{S}^{(h)} \rangle}(z) = \frac{(1+(N-1)z)^t}{(1+z)^{n-k}}$$

Solving strategies

Cost of Gröbner Basis

- **Dense** linear algebra on Macaulay matrix $\boldsymbol{M}_D \rightarrow$ row ech. form
- Cost exponential in D, $2 \leq \omega < 3$:

$$T_{\text{solve}}(\mathcal{S}) = \mathcal{O}(\#\text{cols}(\boldsymbol{M}_D)^{\omega}) = \mathcal{O}\left(\binom{t}{D}^{\omega}(N-1)^{\omega D}\right)$$

Solving degree D from HS

Index of first < 0 coef. in $\mathcal{H}_{R/\langle \mathcal{S}^{(h)} \rangle}$

Hybrid approach I

Conjectured D may be too high to be practical

Hybrid approach (folklore & [BFP10])

Fix $f \geq 0$ variables $+$ solve specialized system $\mathcal{S}_{\text{spec},f}$

Hope: smaller D for $\mathcal{S}_{\text{spec},f}$

$\rightarrow$ Guess $f \geq 0$ zero positions in $\boldsymbol{e}$ (as Prange but $f \ll k$)

- Simplest way: $u \stackrel{def}{=} f/t$ per block, success proba $\left(\frac{\binom{N-1}{u}}{\binom{N}{u}}\right)^t = (1 - u/N)^t$

[BFP10] Bettale, Faugère, and Perret. "Hybrid approach for solving multivariate systems over finite fields".

Hybrid approach II

Cost of solving $\mathcal{S}_{\text{spec},f}$? Same assumptions as for $\mathcal{S}$, same analysis:

$$\mathcal{H}_{R/\langle \mathcal{S}^{(h)}_{\text{spec},f} \rangle}(z) = \frac{(1 + (N - 1 - u)z)^t}{(1 + z)^{n-k}}$$

Final complexity:

$$\mathcal{O}\left(\min_{0 \le u \le N-1} \{(1 - u/N)^{-t} \times T_{\text{solve}}(\mathcal{S}_{\text{spec},u \cdot t})\} \right)$$

- Other ways to fix zeroes (inspired by ISDs ?). We analyze one more in the paper.

Improvement with XL-Wiedemann

Sparse linear algebra. Hope: replace ω by 2

- Kernel of affine Macaulay matrix

Degree D: only highest degree hom. parts

Affine eqs here ! We analyze witness degree d_{wit} [Bar+13, Definition 2]:

- might be strictly larger than D (sometimes by 1 for some parameters)
- still upper bounds from HS machinery !

[Bar+13] Bardet et al. "On the complexity of solving quadratic Boolean systems".

Experimental Verification

We relied on Magma

- Compute HS for both $\mathcal{S}^{(h)}$ and $\mathcal{S}^{(h)}_{\text{spec},f}$ (various f)
- Assumptions regarding d_{wit}: HS again

Conclusion

Conjectured cost with Wiedemann

Parameters from Boyle *et al.* [Boy+19], updated analysis by Liu *et al.* [Liu+22]

Large field: no more $\{\forall i,\ \sum_{j=1}^{N} e_{i,j} = 1\}$, fields eqs of high degree (that's ok)

n	k	t	$\mathbb{F}_2$ [Liu+22]	This work $\mathbb{F}_2$	$\mathbb{F}_{2^{128}}$ [Liu+22]	This work $\mathbb{F}_{2^{128}}$
2^{22}	64770	4788	147	**104**	156	**111**
2^{20}	32771	2467	143	126	155	131
2^{18}	15336	1312	139	123	153	133
2^{16}	7391	667	135	141	151	151
2^{14}	3482	338	132	140	150	152
2^{12}	1589	172	131	136	155	152
2^{10}	652	106	176	**146**	194	180

[Liu+22] Liu et al. *The Hardness of LPN over Any Integer Ring and Field for PCG Applications.*

More on the results

- Sometimes beats Gauss/ISDs for very low rates ("Primal")
- Zone with constant deg. $D \rightarrow$ polynomial algorithm ?

Rather similar to Arora-Gê on LWE [AG11]
(polynomial for sufficiently many samples)

[AG11] Arora and Ge. "New Algorithms for Learning in Presence of Errors". *Automata, Languages and Programming*.