Polynomial systems, sparsity, and applications

Matías R. Bender

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Short CV

- Since 2019, PostDoc at Institut für Mathematik, TU Berlin.
  (Mentor: P. Bürgisser)
- In 2019, PhD in Informatics from Sorbonne Université.
  (Advisors: J.-C. Faugère & E. Tsigaridas)
- In 2015, Bsc+Msc in Computer Science from Univ. de Buenos Aires.

Supported by the ERC under the European's Horizon 2020 research and innovation programme (grant no. 787840)

It is all about the balance

	Model	Computation
Linear algebra	Linear	Easy
Non-linear algebra	Non-linear	Hard

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Pragmatic approach $\rightarrow$ Use more powerful models,
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I study how to exploit ther inputs' structure to compute faster.

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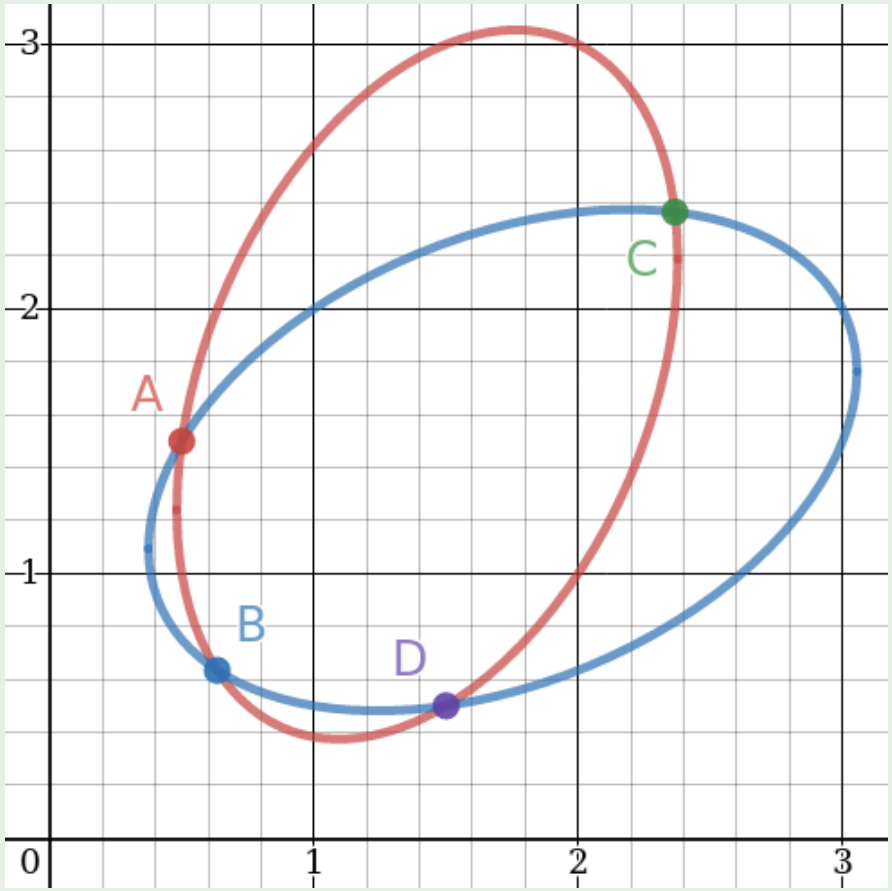
Solving structured polynomial systems

- Solving polynomial systems of degree at most d in $\mathbb{C}[x_1, \ldots, x_n]$.



$$\begin{cases} 2x^2 - xy + y^2 - 4x - 2y + 3 = 0 \\ x^2 - xy + 2y^2 - 2x - 4y + 3 = 0 \end{cases}$$

$$\boxed{\begin{array}{c} d = 2 \\ n = 2 \end{array}}$$

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Applications

Tensor decomposition, topological data analysis, computational biology.

Outline of the talk

1. Classical tools to solve dense polynomial systems
   - Gröbner bases
   - Resultants
2. Sparse polynomials
3. Solving sparse systems
   - Gröbner bases
   - Numerical linear algebra approaches
   - Complete intersections over toric varieties
   - Determinantal formulas for mixed multilinear systems
4. Applications
5. Future work

Gröbner basics

- $\mathbb{C}[\boldsymbol{x}] = \mathbb{C}[x_1, \ldots, x_n]$, polynomial ring in n indeterminates over $\mathbb{C}$.
- Polynomial $\rightarrow \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \in \mathbb{C}[\boldsymbol{x}]$.
- Monomial $\rightarrow \boldsymbol{x}^\alpha$, for $\alpha \in \mathbb{N}^n$.

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  $\rightarrow$ set of generators $(g_1, \ldots, g_s)$ with special properties.

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It depends on a monomial ordering.

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  - Ideal membership: Given h', there exists $h_1, \ldots, h_r$ such $h' = \sum_i h_i f_i$?

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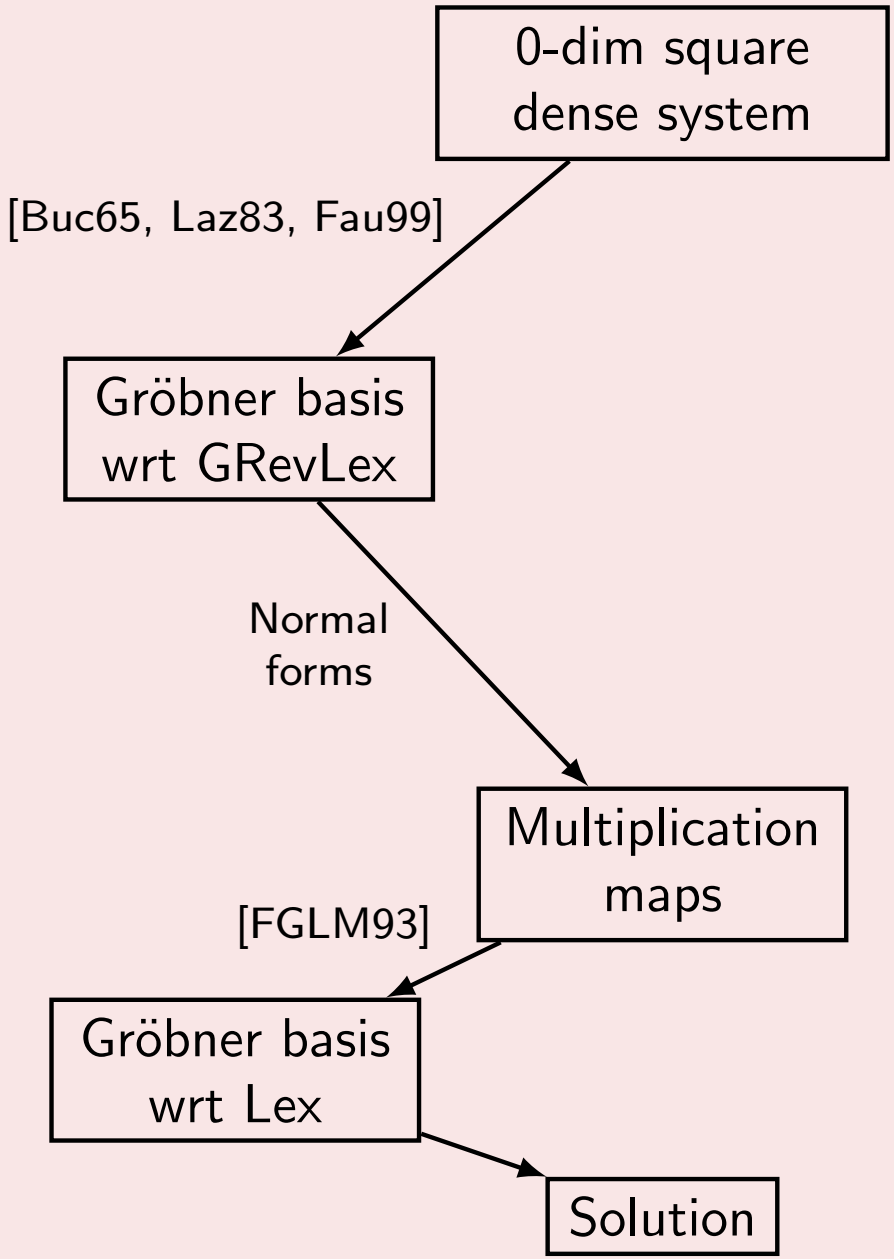
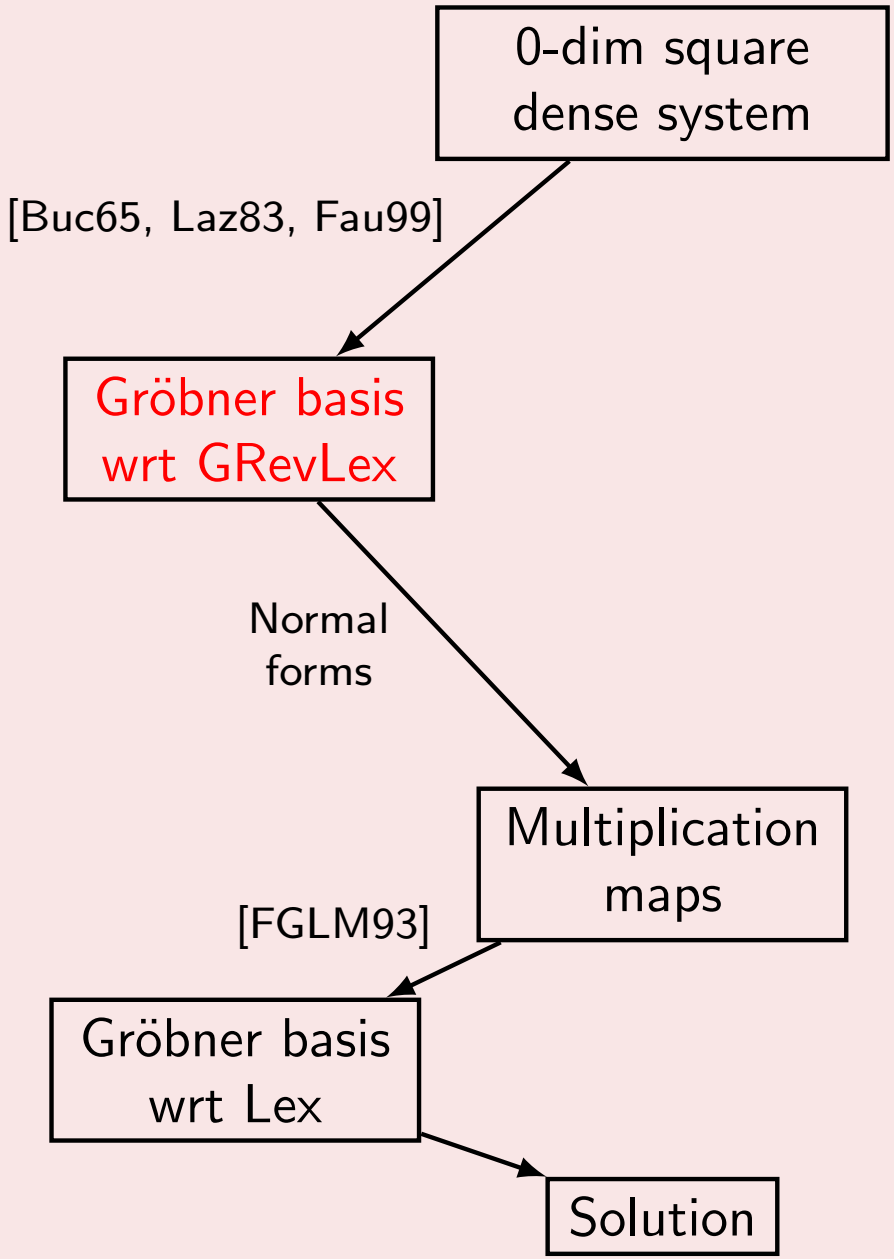
  - Solve: $\{p \in \mathbb{C}^n : f_1(p) = \cdots = f_r(p) = 0\}$.
  - Compute other algebraic and geometric invariants.



Under suitable assumptions



Under suitable assumptions

Computing Gröbner bases - [Lazard '83]

$(f_1, f_2, \ldots, f_n)$ - - - - - - ➤ Gröbner basis

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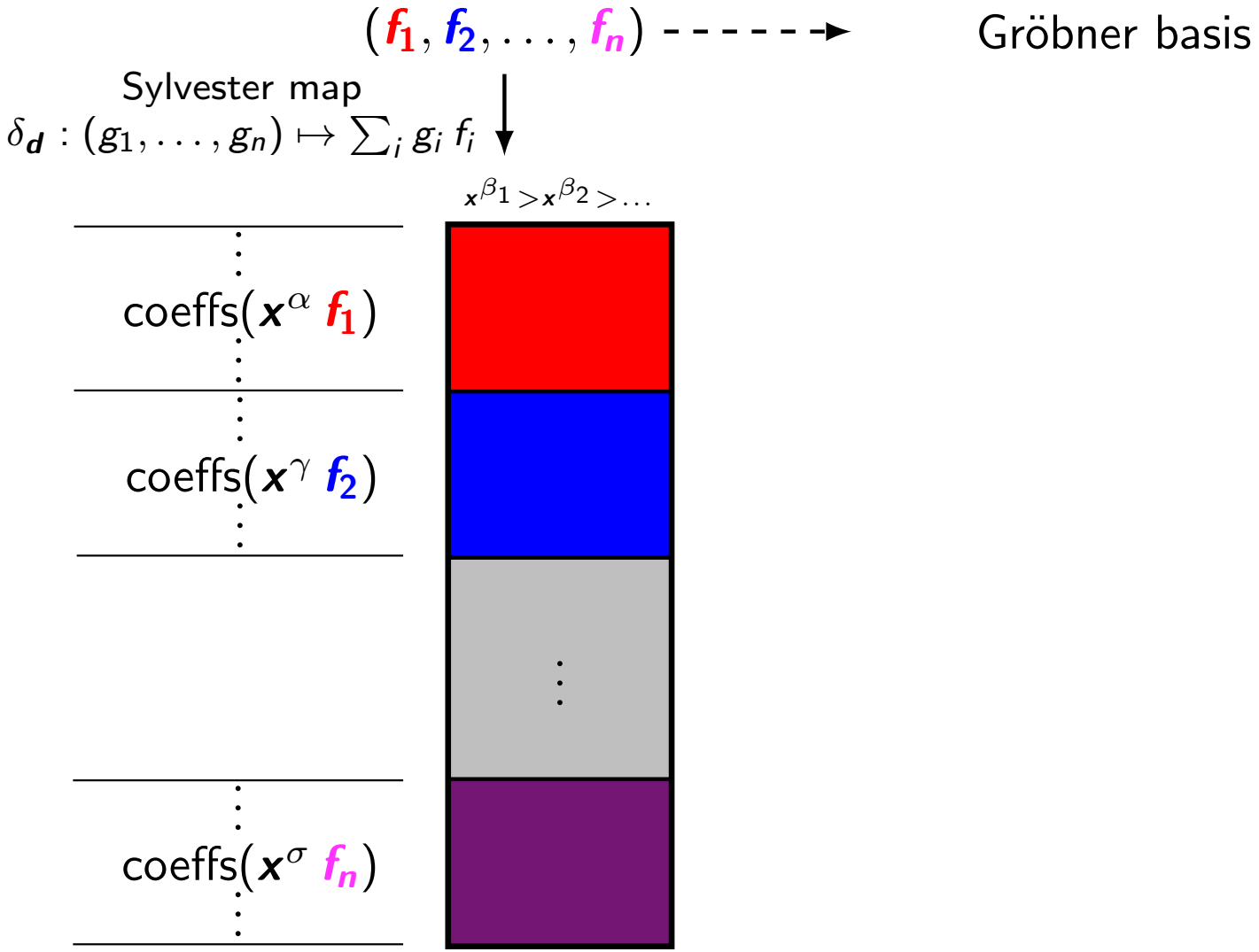

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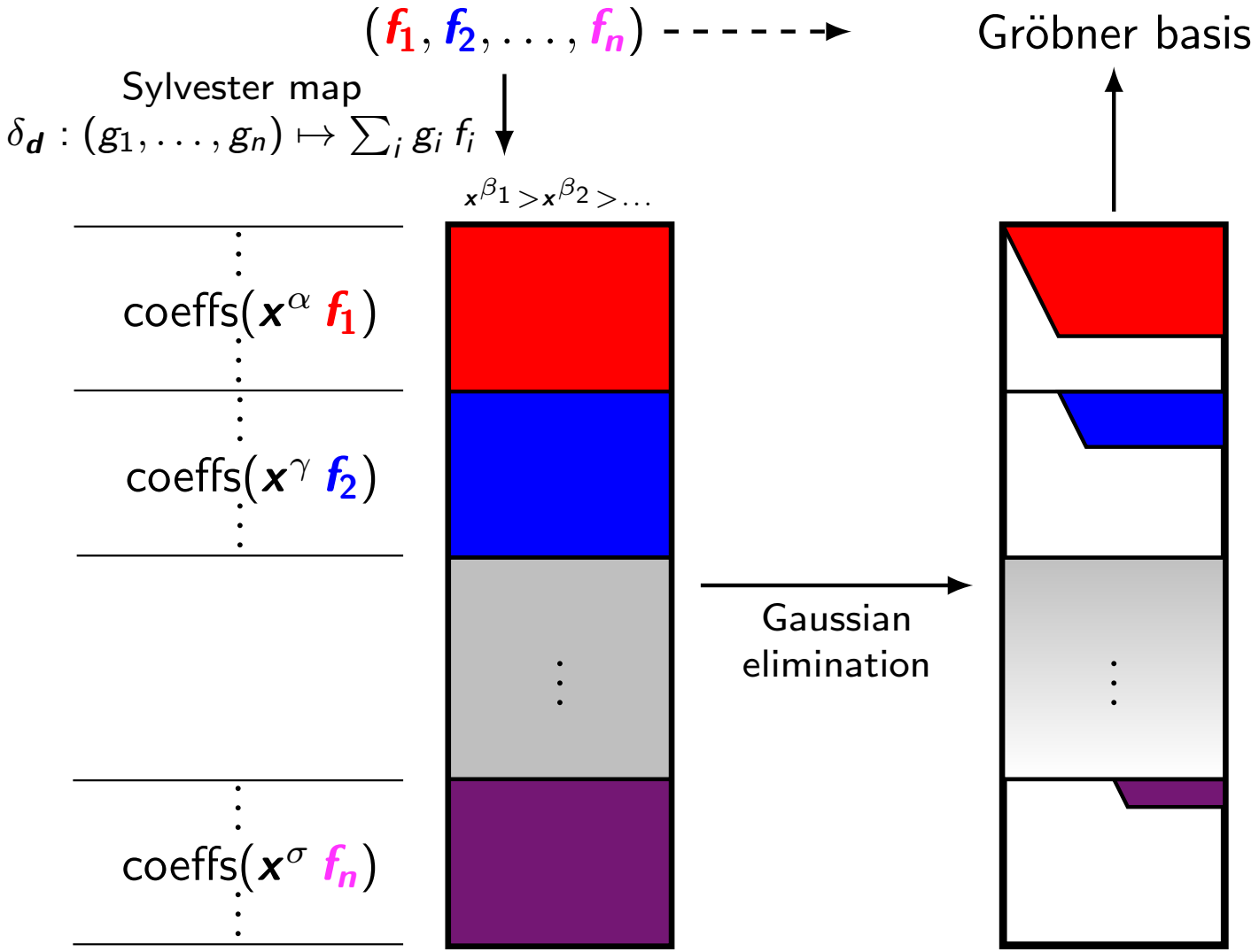

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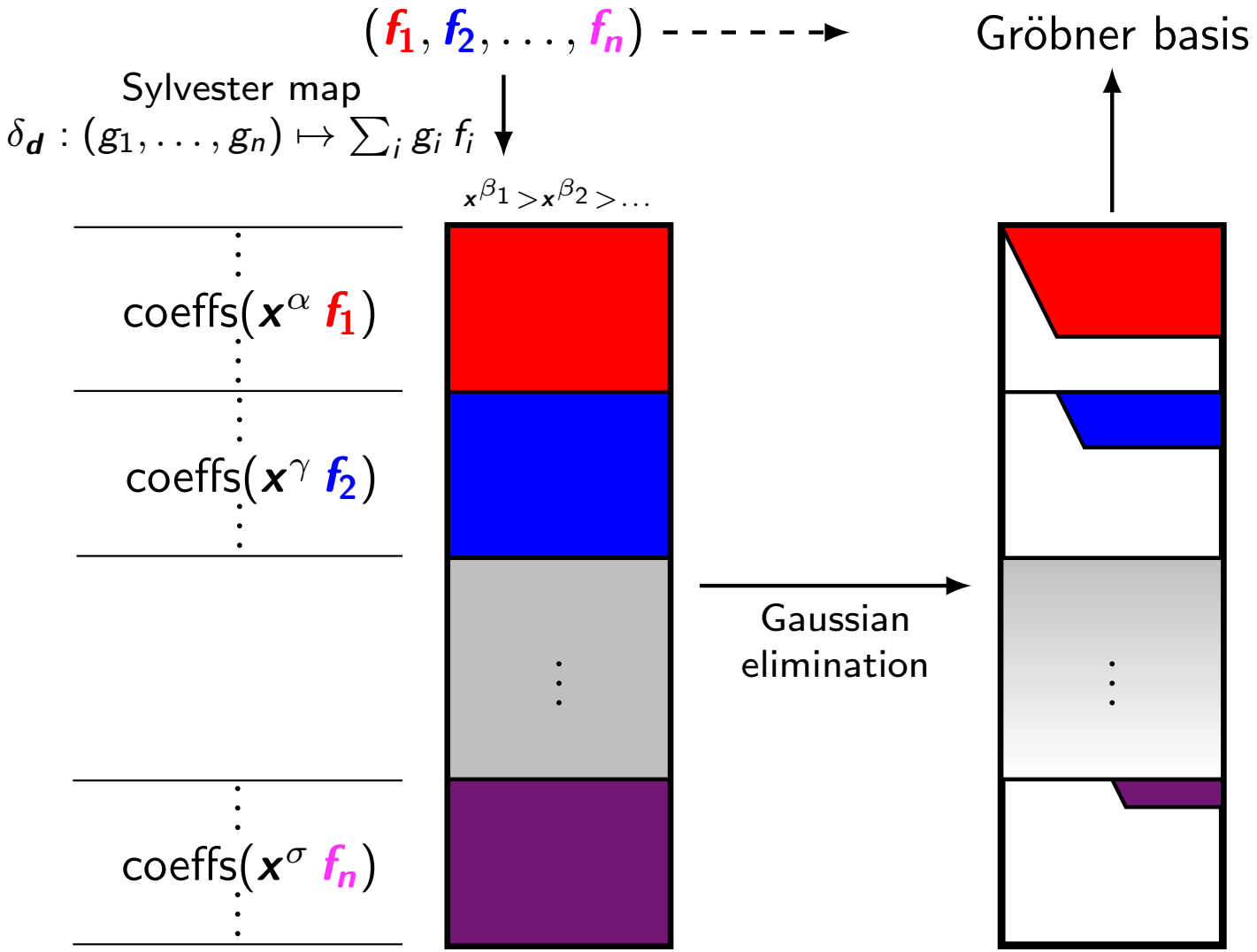

Complexity $\leftrightarrow$ Size of matrix $\rightarrow$ In generic coord., $d \leq$ Castelnuovo-Mumford (CM) reg.
[Bayer, Stillman '87]

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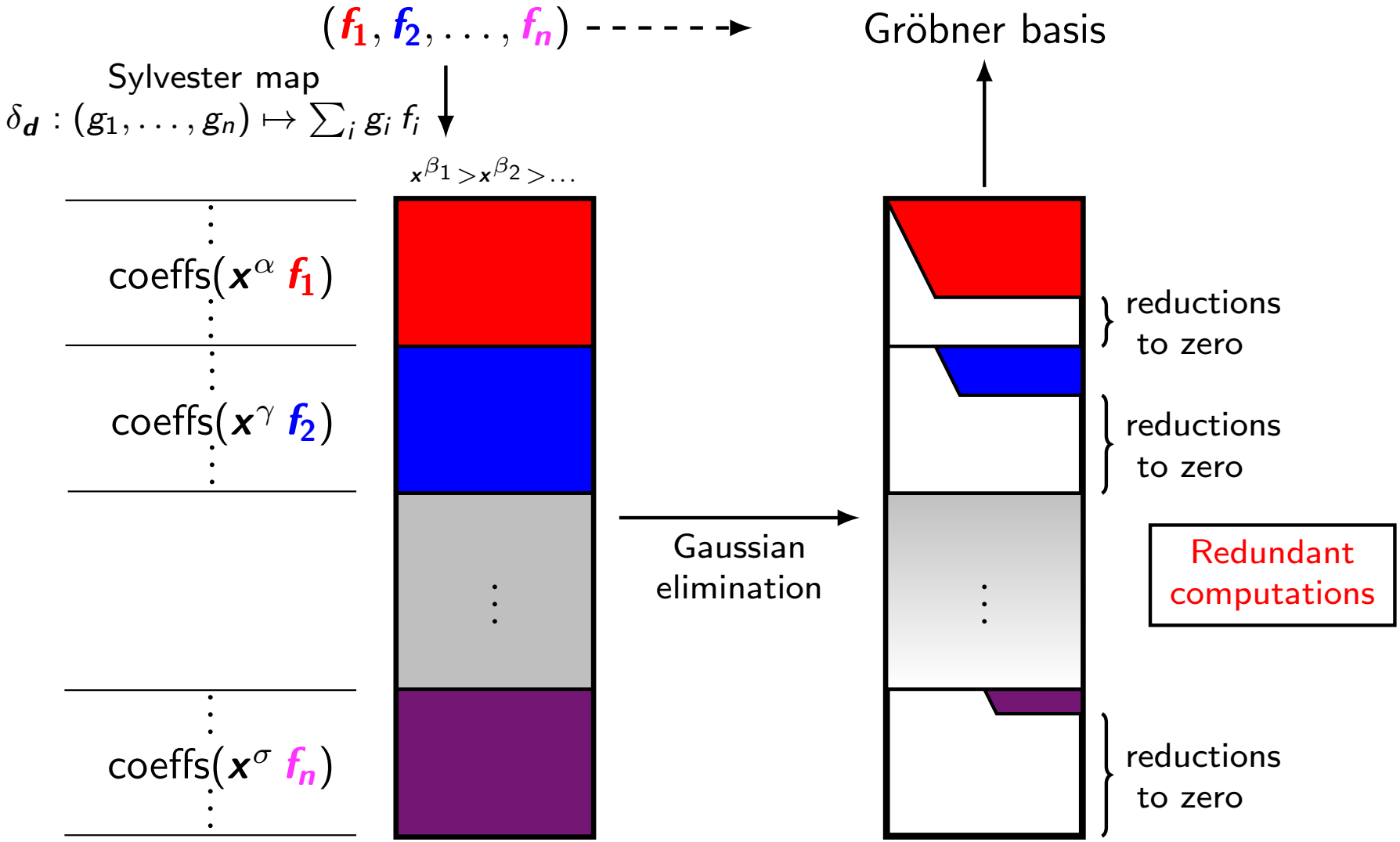

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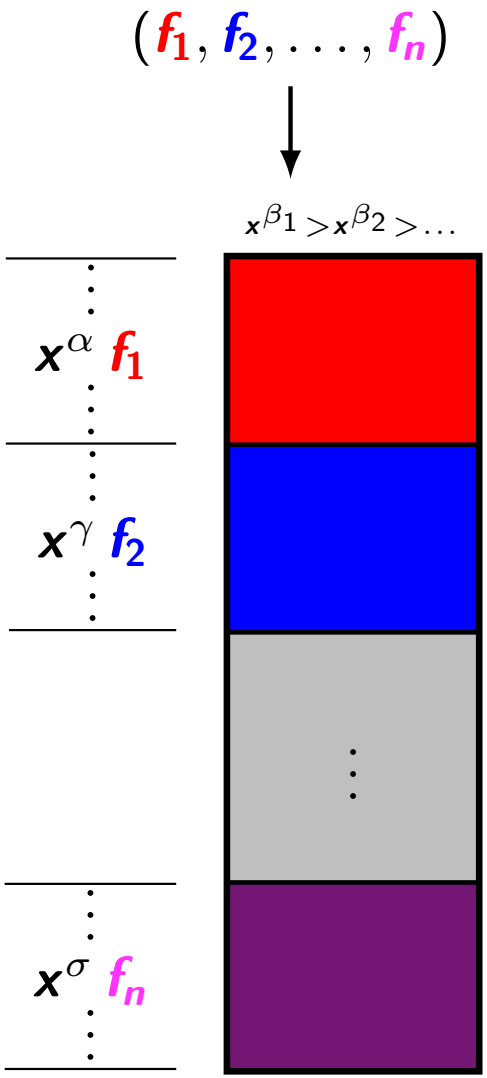

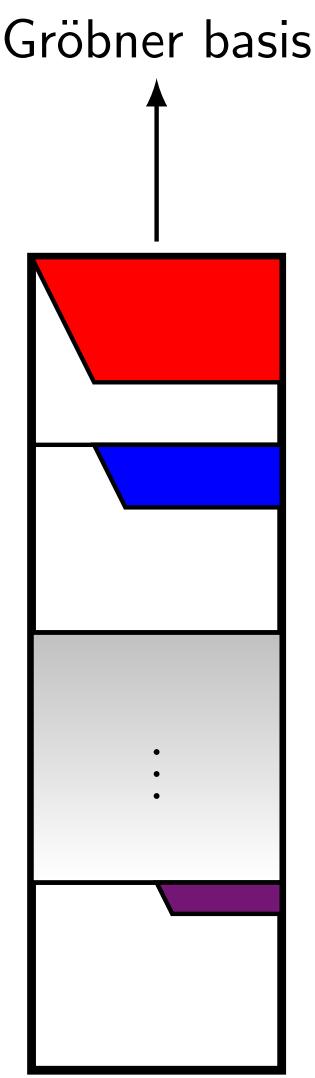

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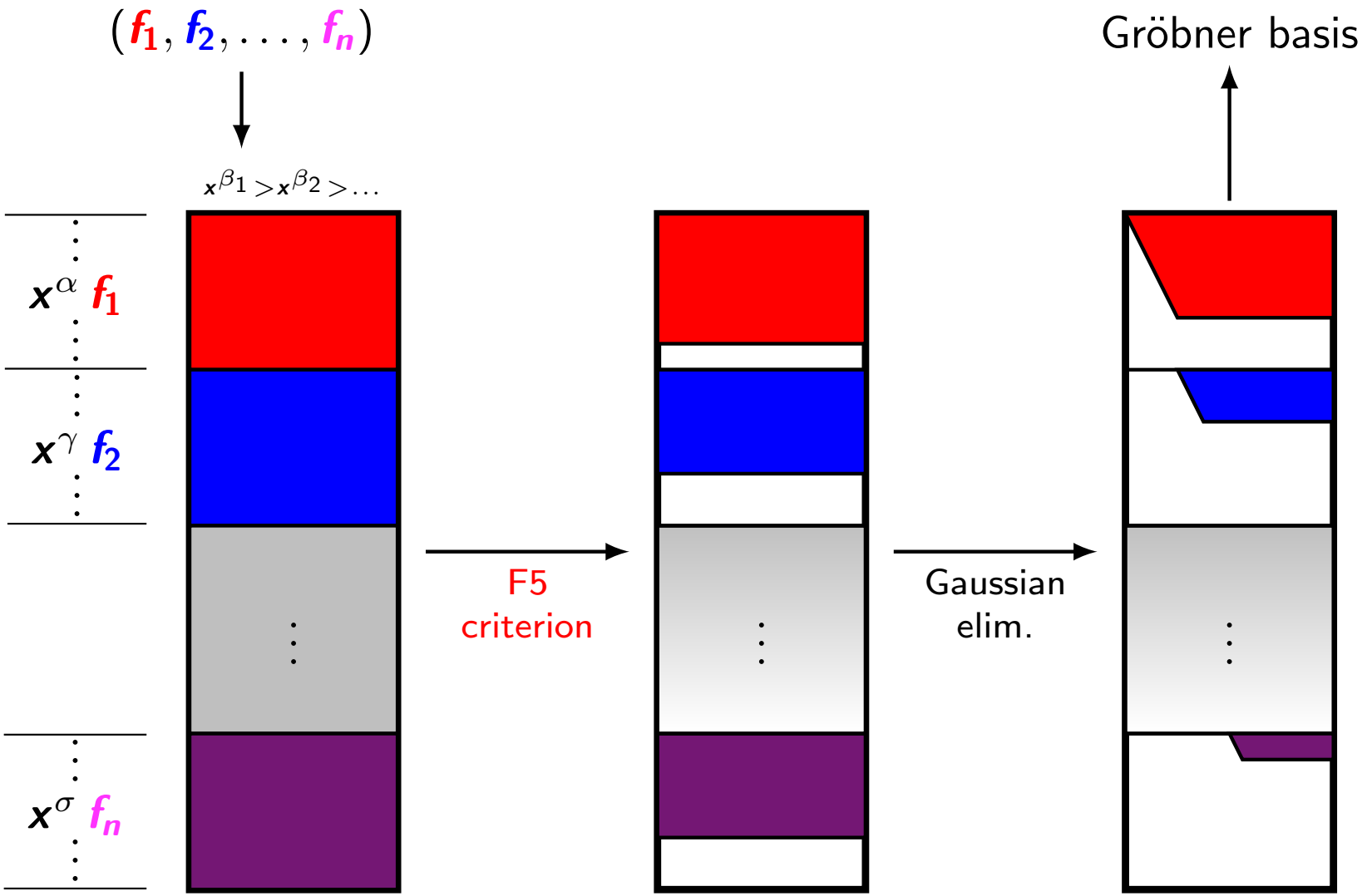

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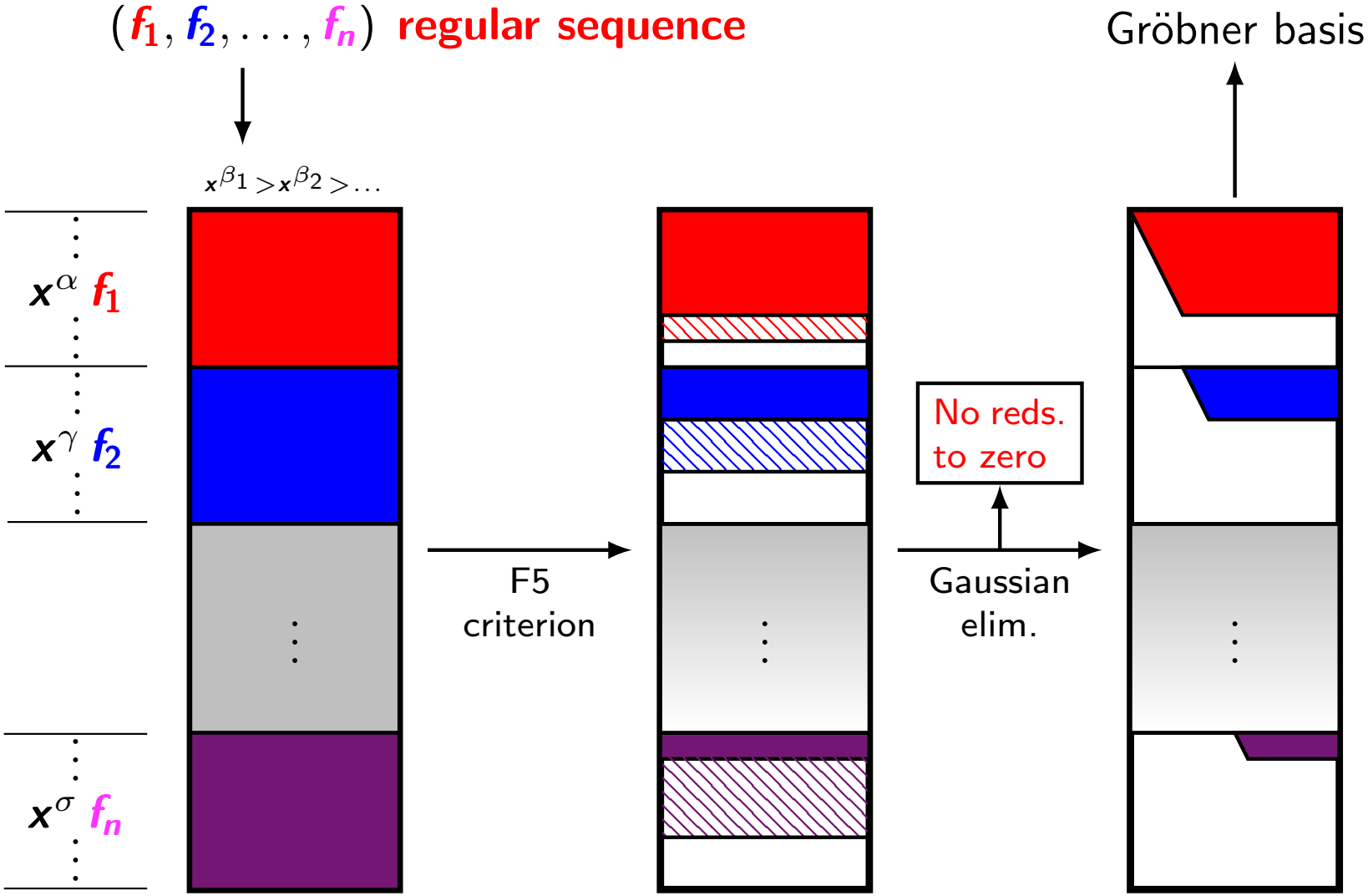

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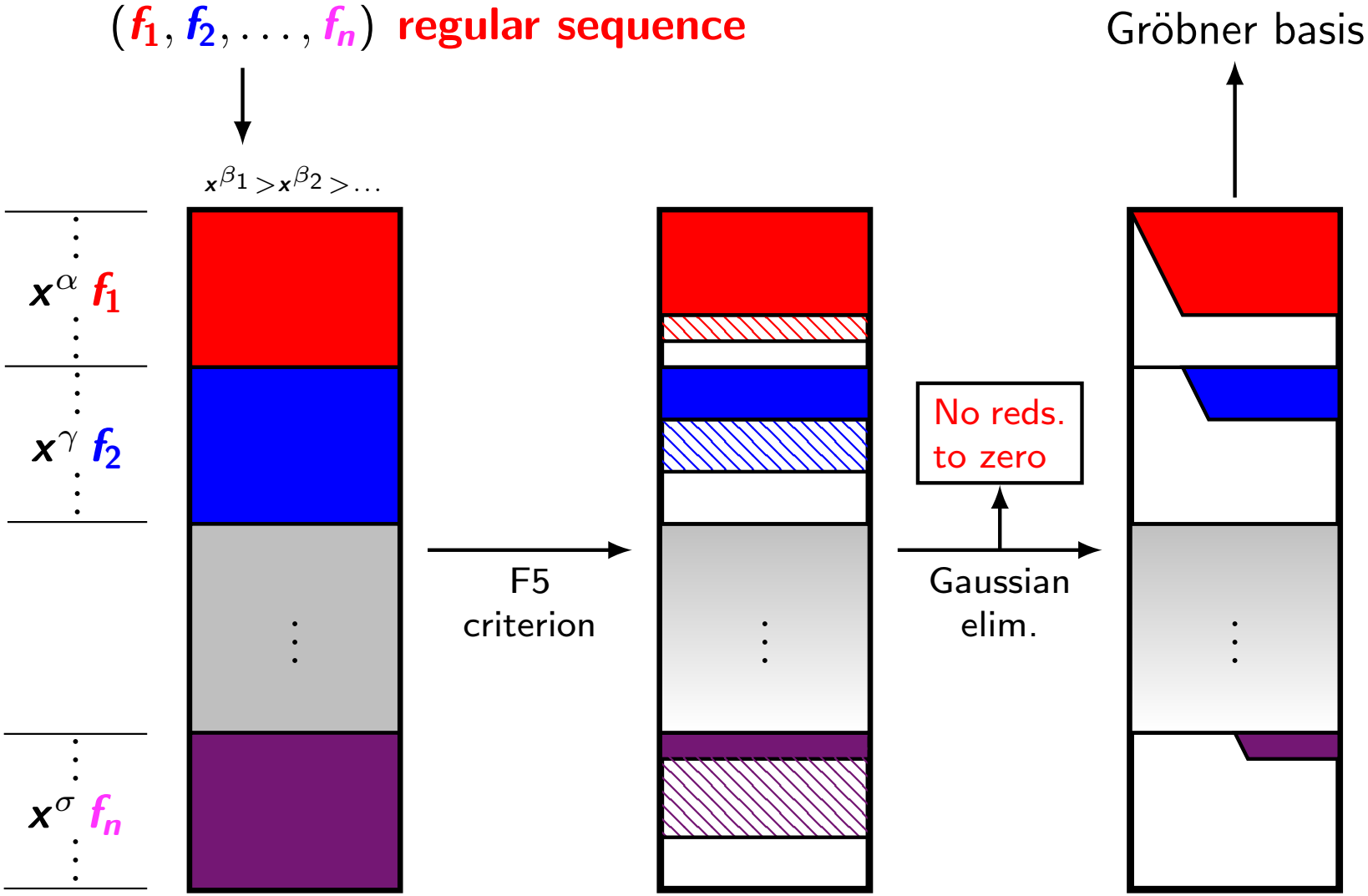

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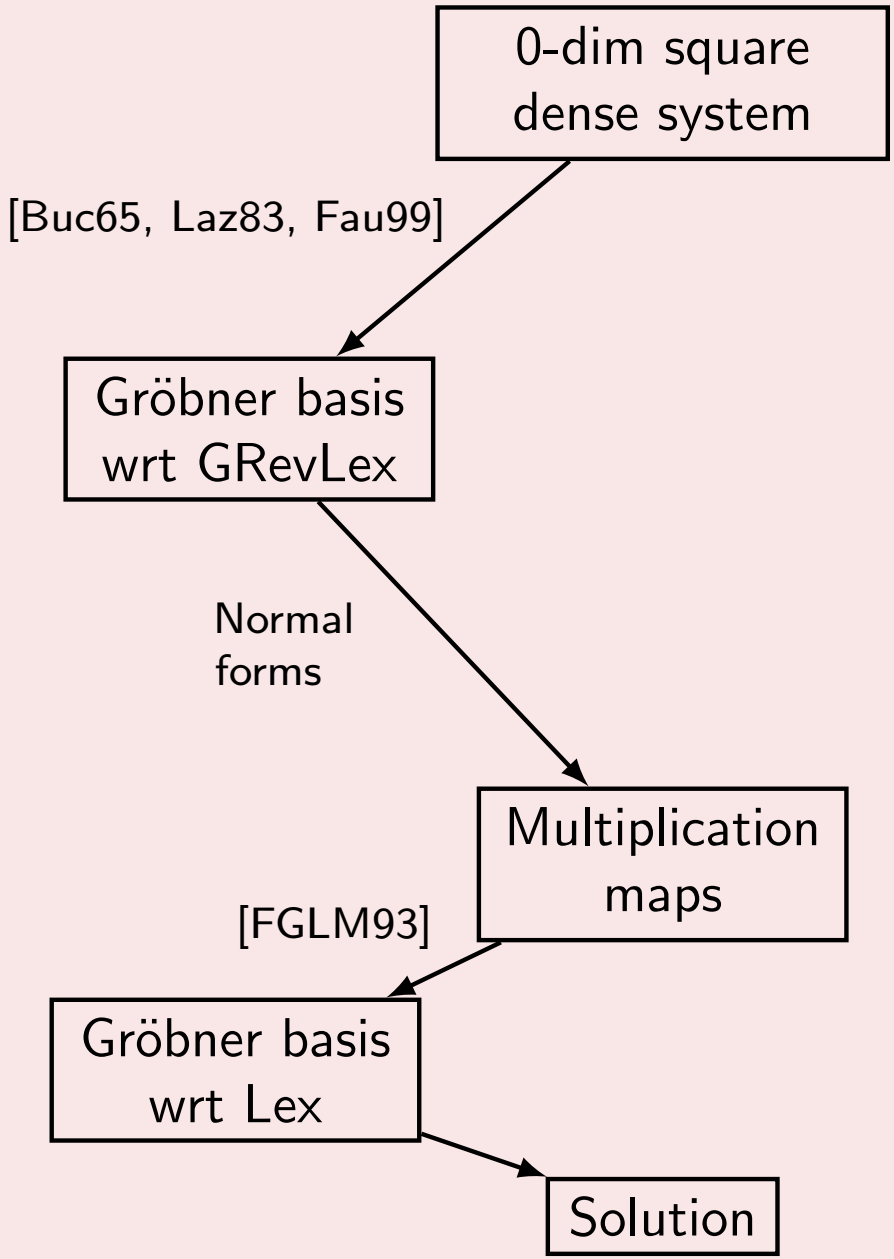
$\rightarrow$ Macaulay bound: CM reg. $= \sum_{i=1}^{n} \deg(f_i) - n + 1$



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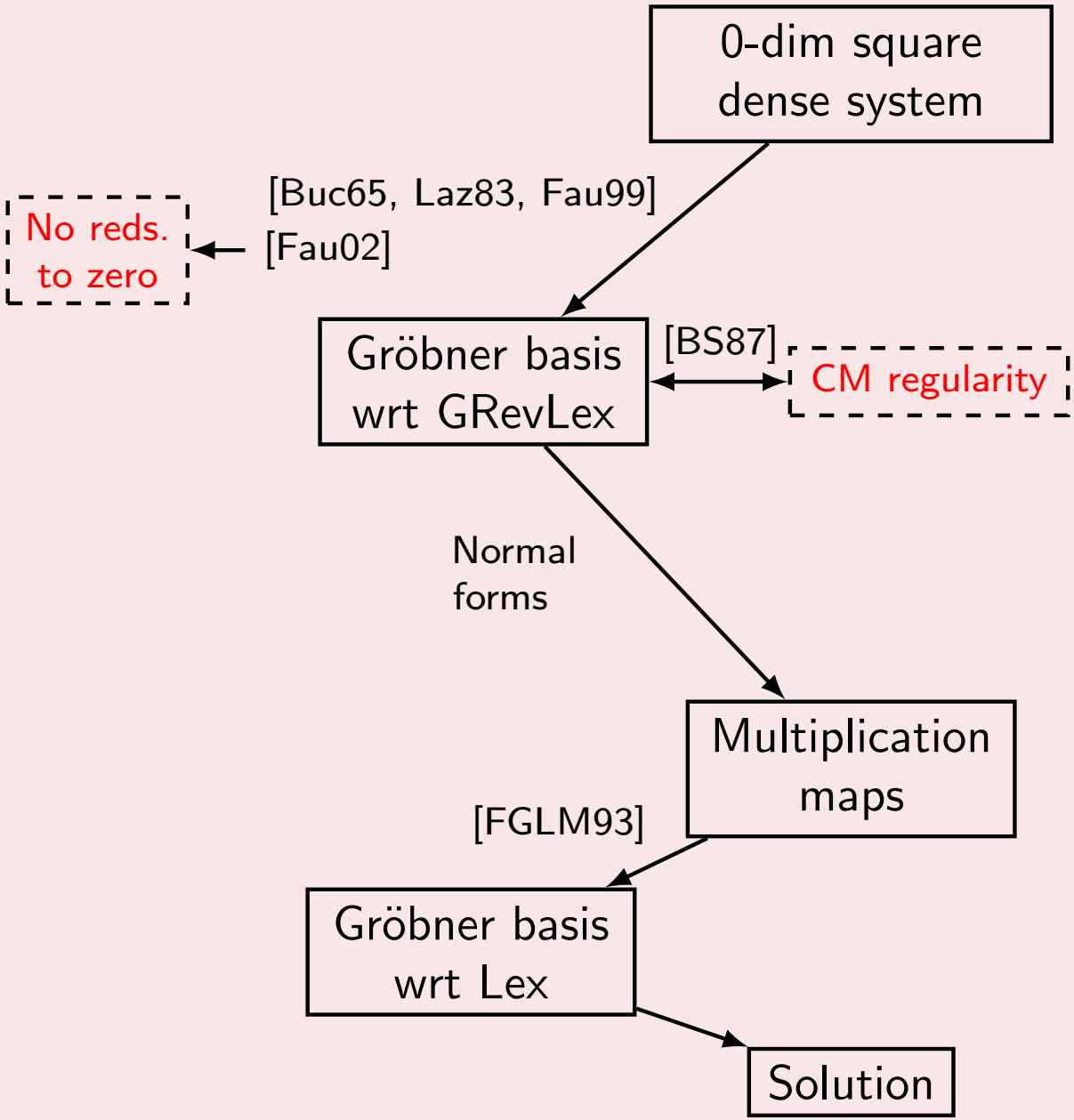

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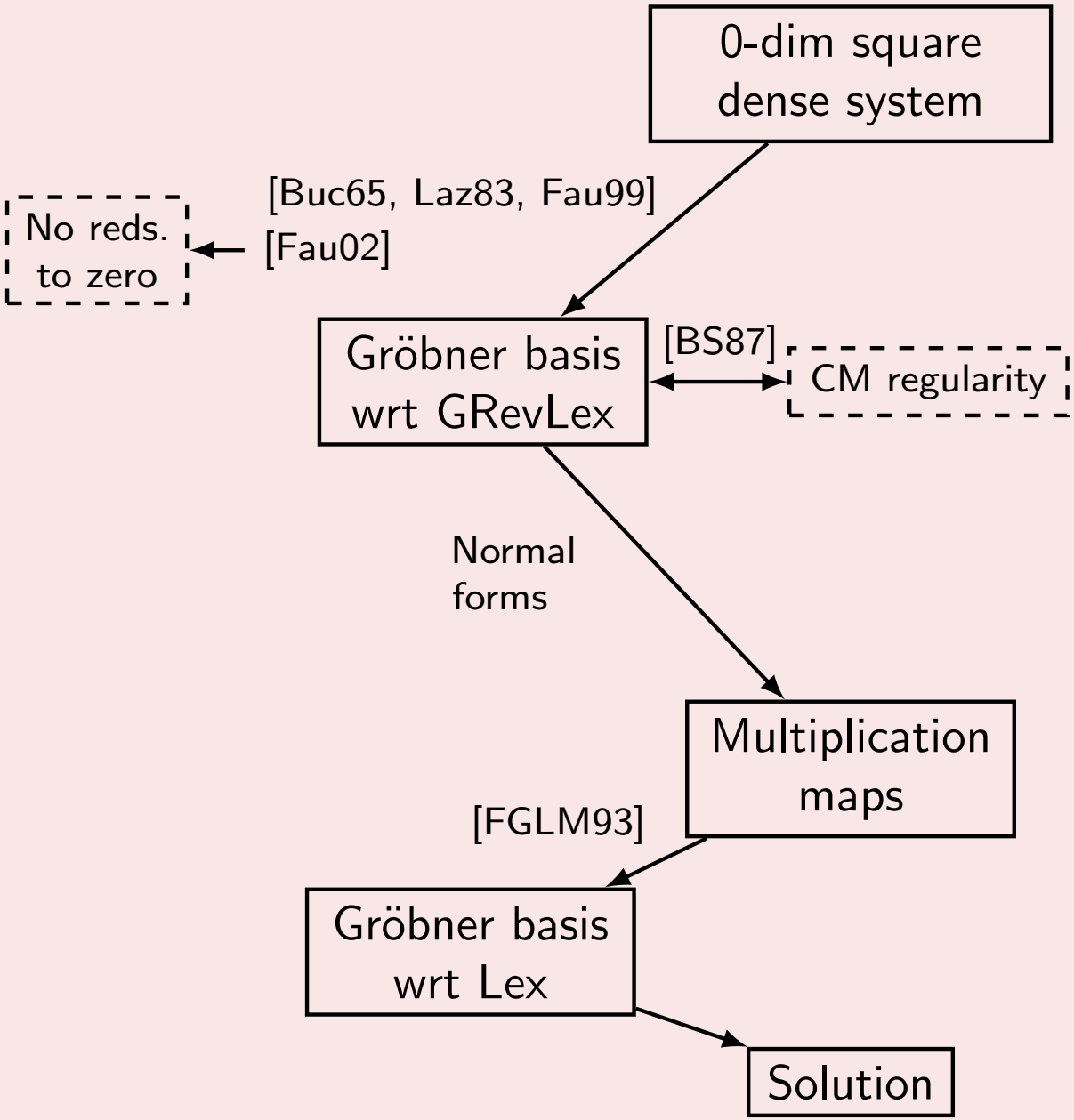

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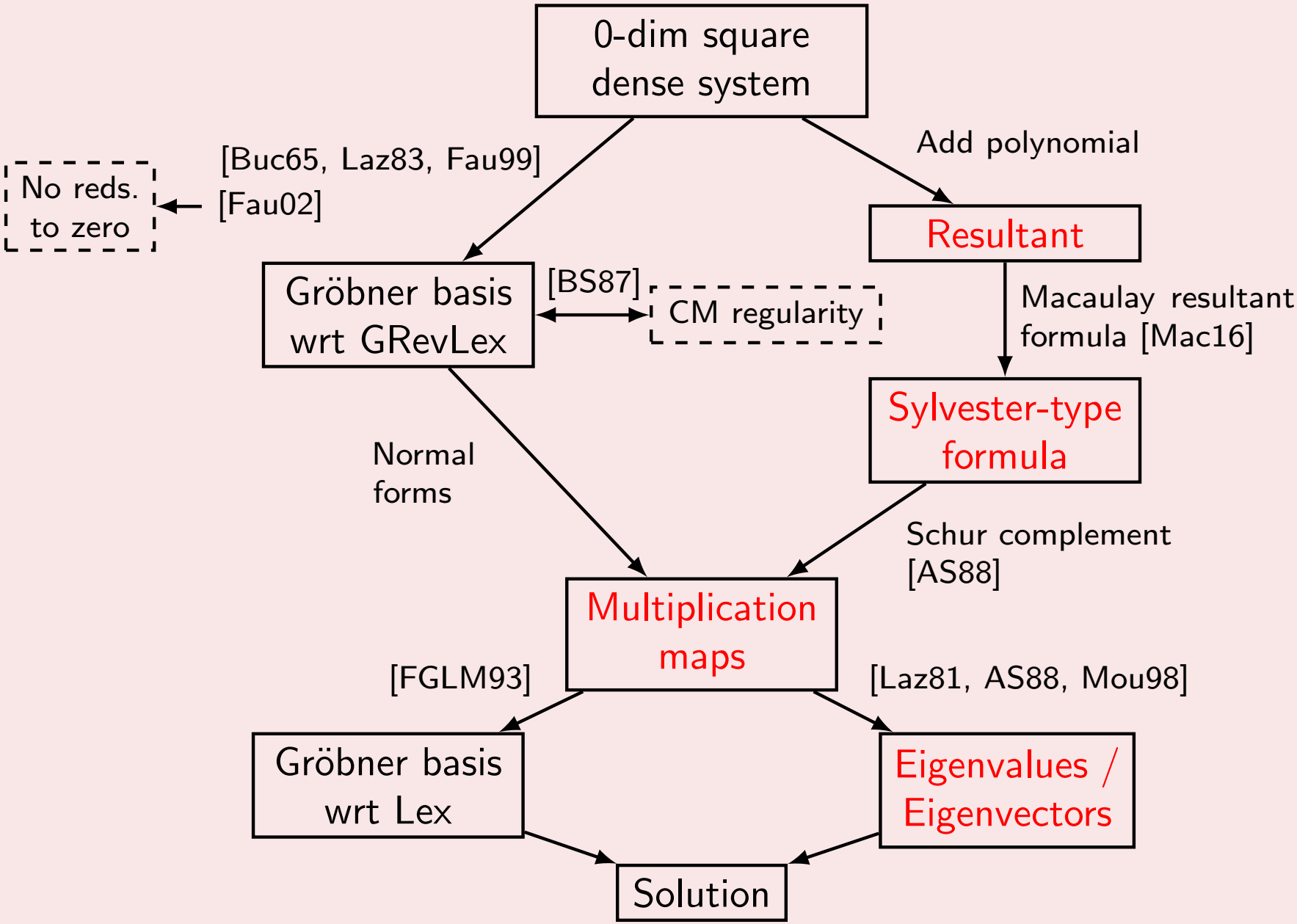

Under suitable assumptions

The resultant and Sylvester-type formulas

Projective resultant

Necessary and sufficient condition for a homogeneous system in $(f_0, \ldots, f_n) \in \mathbb{C}[x_0, \ldots, x_n]^{n+1}$ to have solutions in $\mathbb{P}^n$.

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Example : Resultant of linear forms = Determinant

$$\begin{cases} a_1\, x + a_2\, y + a_3\, z = 0 \\ b_1\, x + b_2\, y + b_3\, z = 0 \\ c_1\, x + c_2\, y + c_3\, z = 0 \end{cases} \quad \text{has a solution over } \mathbb{P}^2$$

$$\Updownarrow$$

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 0.$$

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Classical way of computing resultant $\rightarrow$ Sylvester-type formula

$$(g_0, \ldots, g_n) \mapsto \sum_{i=0}^{n} g_i \, f_i$$

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Macaulay resultant matrix [Macaulay, 1916]

$$\begin{cases} f_1 := & a_1 x^2 + a_2 x y + a_3 x z + \\ & a_4 y^2 + a_5 y z + a_6 z^2 \\ f_2 := & b_1 x + b_2 y + b_3 z \\ f_3 := & c_1 x + c_2 y + c_3 z \end{cases}$$

	x^2	$x\,y$	$x\,z$	y^2	$y\,z$	z^2
f_1	a_1	a_2	a_3	a_4	a_5	a_6
$x\,f_2$	b_1	b_2	b_3			
$y\,f_2$		b_1		b_2	b_3	
$z\,f_2$			b_1		b_2	b_2
$y\,f_3$		c_1		c_2	c_3	
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$$\texttt{Determinant} = \texttt{Resultant} \cdot \texttt{ExtraFactor}.$$

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$$\texttt{Determinant} = \texttt{Resultant} \cdot \texttt{ExtraFactor}.$$

Determinantal formula $\rightarrow$ `ExtraFactor` is a constant.

Solving via Sylvester-type formulas

- We want to compute the two solutions $\alpha, \beta \in \mathbb{C}^2$ of (f_1, f_2)

$$\begin{cases} f_1 := & 1\,x^2 + -1\,x\,y + 4\,x + -2\,y^2 + -5\,y + 3 \\ f_2 := & 1\,x + -1\,y + -1 \end{cases}.$$

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- Introduce $f_3 := -1\,x + 2\,y + 1$ and consider a Sylvester-type formula.

$$\left(\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right) =$$

	x^2	$x\,y$	x	y^2	y	1
f_1	1	−1	4	−2	−5	3
$x\,f_2$	1	−1	−1			
$y\,f_2$		1		−1	−1	
f_2			1		−1	−1
$y\,f_3$		−1		2	1	
f_3			−1		2	1

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$$\left(\begin{array}{c|c} M_{1,1} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} \end{array}\right) =$$

	x^2	$x\,y$	x	y^2	y	1
f_1	1	−1	4	−2	−5	3
$x\,f_2$	1	−1	−1			
$y\,f_2$		1		−1	−1	
f_2			1		−1	−1
$y\,f_3$		−1		2	1	
f_3			−1		2	1

- Schur complement of $M_{2,2}$ $\leftrightarrow$ Multiplication map of f_3 in $\mathbb{C}[x,y]/\langle f_1, f_2\rangle$

$$\widetilde{M}_{2,2} = M_{2,2} - M_{2,1}\,M_{1,1}^{-1}\,M_{1,2} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

Solving via Sylvester-type formulas

- We want to compute the two solutions $\alpha, \beta \in \mathbb{C}^2$ of (f_1, f_2)

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- Eigenvalues of $\widetilde{M}_{2,2}$ [Lazard, 1981]

$$f_3(\boldsymbol{\alpha}) = 2 \qquad \text{and} \qquad f_3(\boldsymbol{\beta}) = -2.$$

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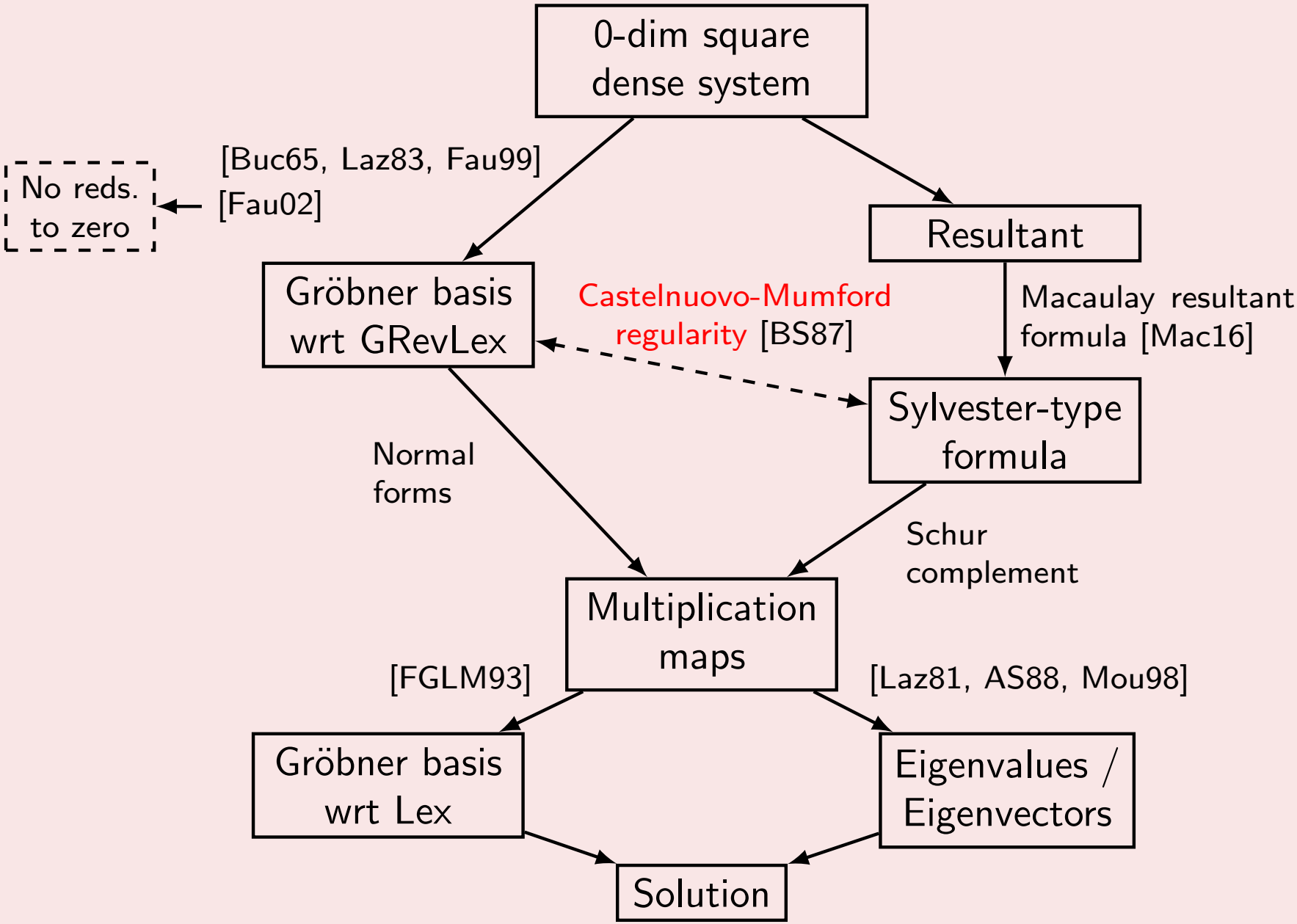
- Eigenvectors of $\widetilde{M}_{2,2}$ [Auzinger & Stetter, 1988]

$$\begin{pmatrix} \alpha_y \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \beta_y \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

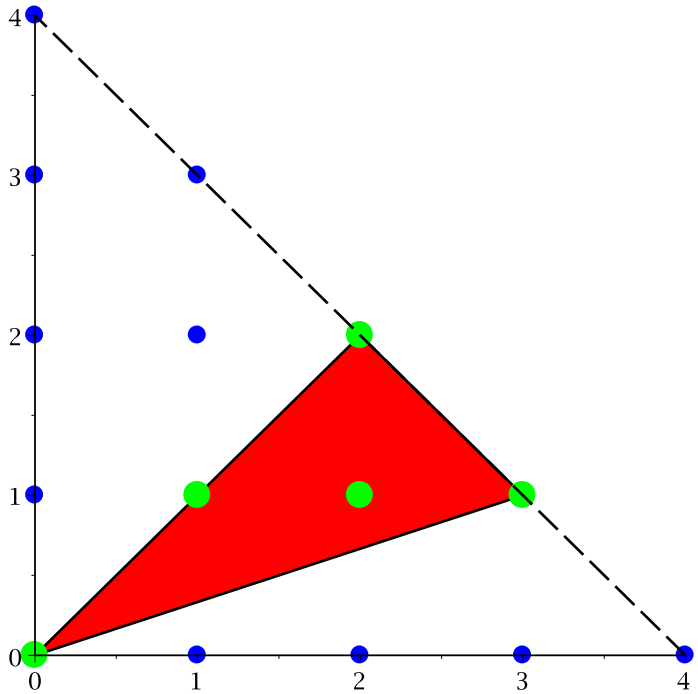

0-dim square dense system
[Buc65, Laz83, Fau99]
No reds. to zero
[Fau02]
Resultant
Gröbner basis wrt GRevLex
Castelnuovo-Mumford regularity [BS87]
Macaulay resultant formula [Mac16]
Sylvester-type formula
Normal forms
Schur complement
Multiplication maps
[FGLM93]
[Laz81, AS88, Mou98]
Gröbner basis wrt Lex
Eigenvalues / Eigenvectors
Solution
Under suitable assumptions

Sparse polynomials / systems

Sparse polynomials / systems

- **Newton polytope** of $f = \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \longrightarrow$ Convex hull of $\{\alpha : c_\alpha \neq 0\}$.
- Sparse polynomial $\longrightarrow$ Its Newton polytope is "small".

$$1 + xy + x^2y + x^2y^2 + x^3y = \mathbf{1} + \mathbf{0} \cdot x + \mathbf{0} \cdot y + \mathbf{0} \cdot x^2 + \boldsymbol{xy} + \mathbf{0} \cdot y^2 + \mathbf{0} \cdot x^3 + \boldsymbol{x^2y} + \mathbf{0} \cdot xy^2 + \mathbf{0} \cdot y^3 + \mathbf{0} \cdot x^4 + \boldsymbol{x^3y} + \boldsymbol{x^2y^2} + \mathbf{0} \cdot xy^3 + \mathbf{0} \cdot y^4$$



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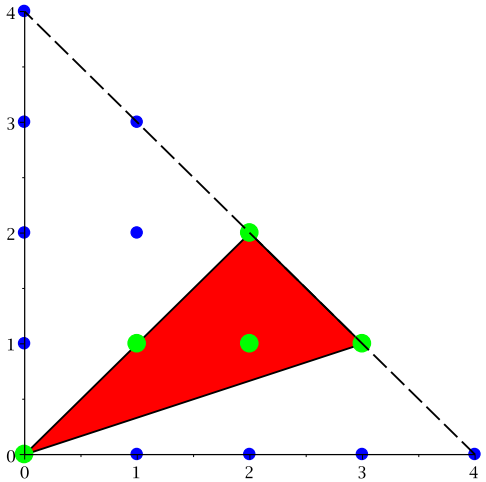
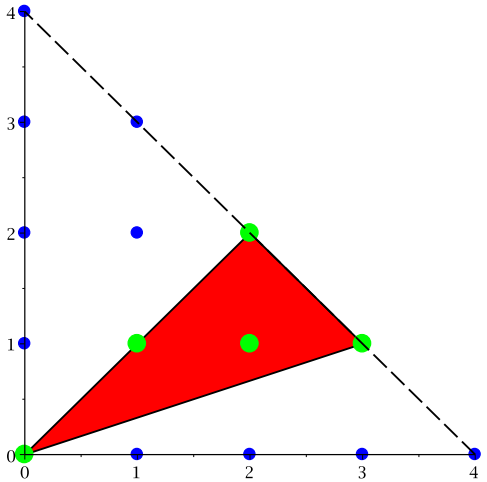
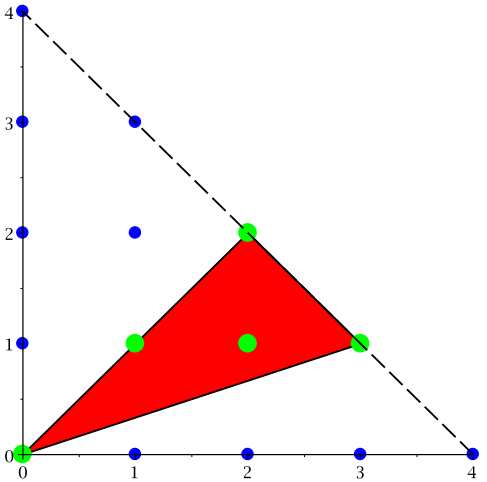
$f_1 := 1 + xy + x^2y + x^2y^2 + x^3y$



$f_2 := 3 - xy + x^2y - x^2y^2 + x^3y$



$f_3 := 2 + xy - x^2y + x^2y^2 + x^3y$



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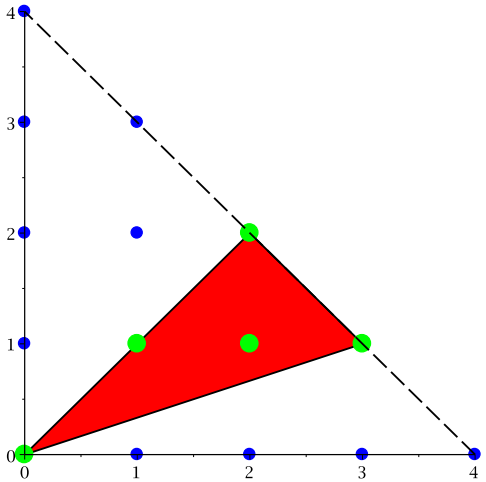
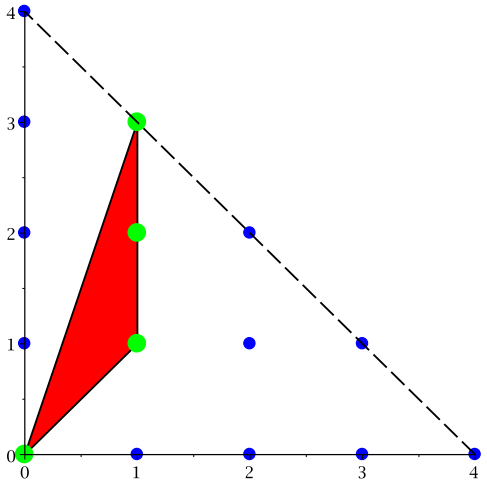
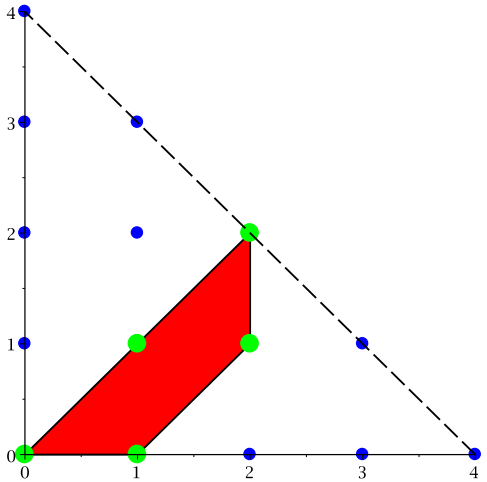


$f_2 := 1 + xy + xy^2 + xy^3$



$f_3 := 1 + x + xy + x^2y + x^2y^2$



Exploiting sparsity [Gelfand, Kapranov, & Zelevinsky '90]

- Sparse resultant $\rightarrow$ Generalization of the resultant for sparse systems.

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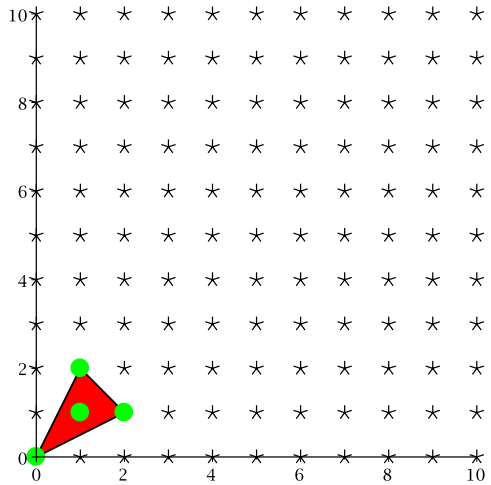

$$1 + x\,y + x^2\,y + x\,y^2 \in \qquad \mathbb{C}[x, y]$$

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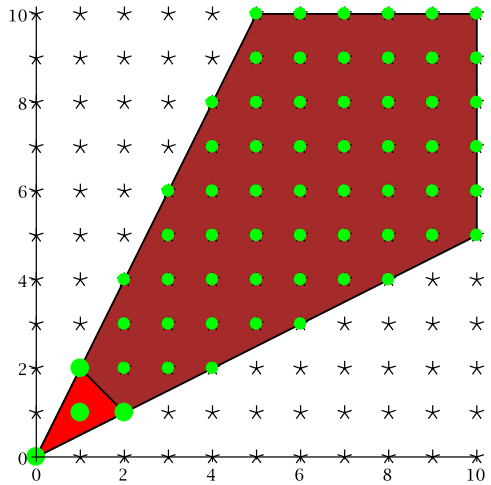

$$1 + x\,y + x^2\,y + x\,y^2 \in \mathbb{C}[x\,y, x^2\,y, x\,y^2] \subset \mathbb{C}[x, y]$$

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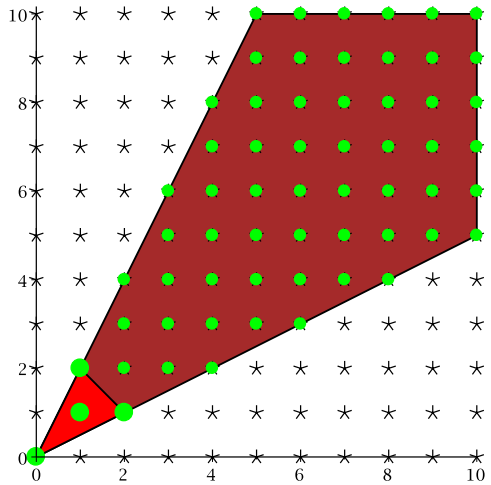

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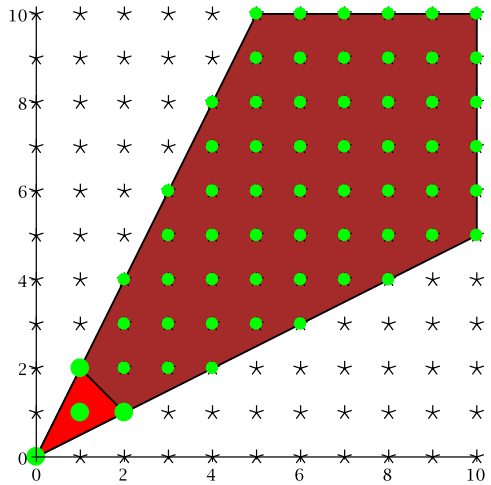

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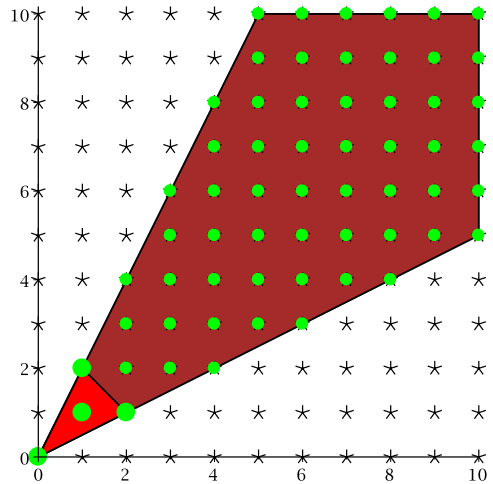

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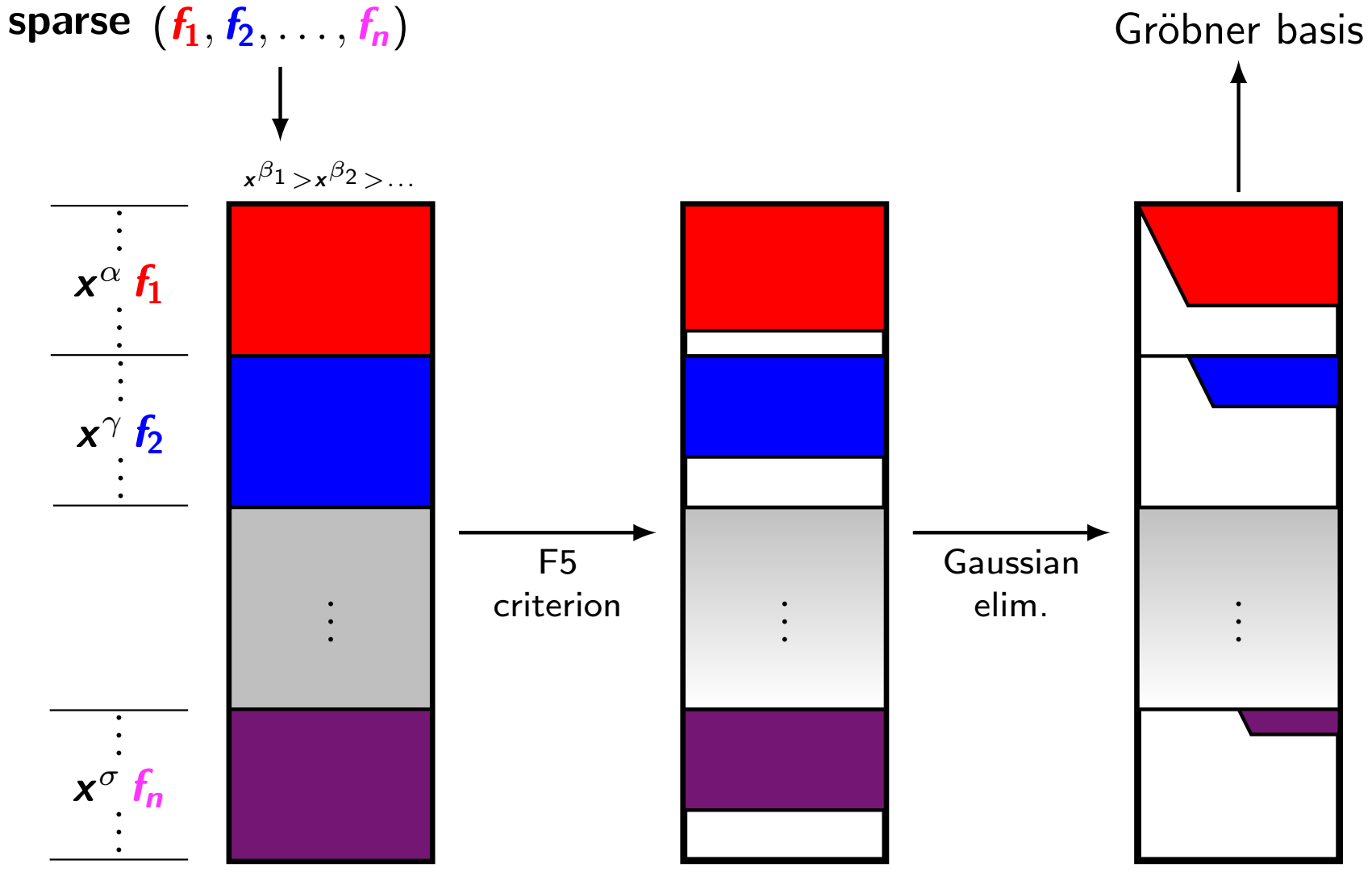
  - **Weighted projective space** $\mathbb{P}(w_0, \ldots, w_n)$.
    $\rightarrow$ Weighted homogeneous polynomials $\deg(x_i) = w_i$.

Gröbner bases for sparse systems

Joint work with Jean-Charles Faugère & Elias Tsigaridas.

Computing Gröbner bases for sparse systems



Complexity $\leftrightarrow$ Size of matrix

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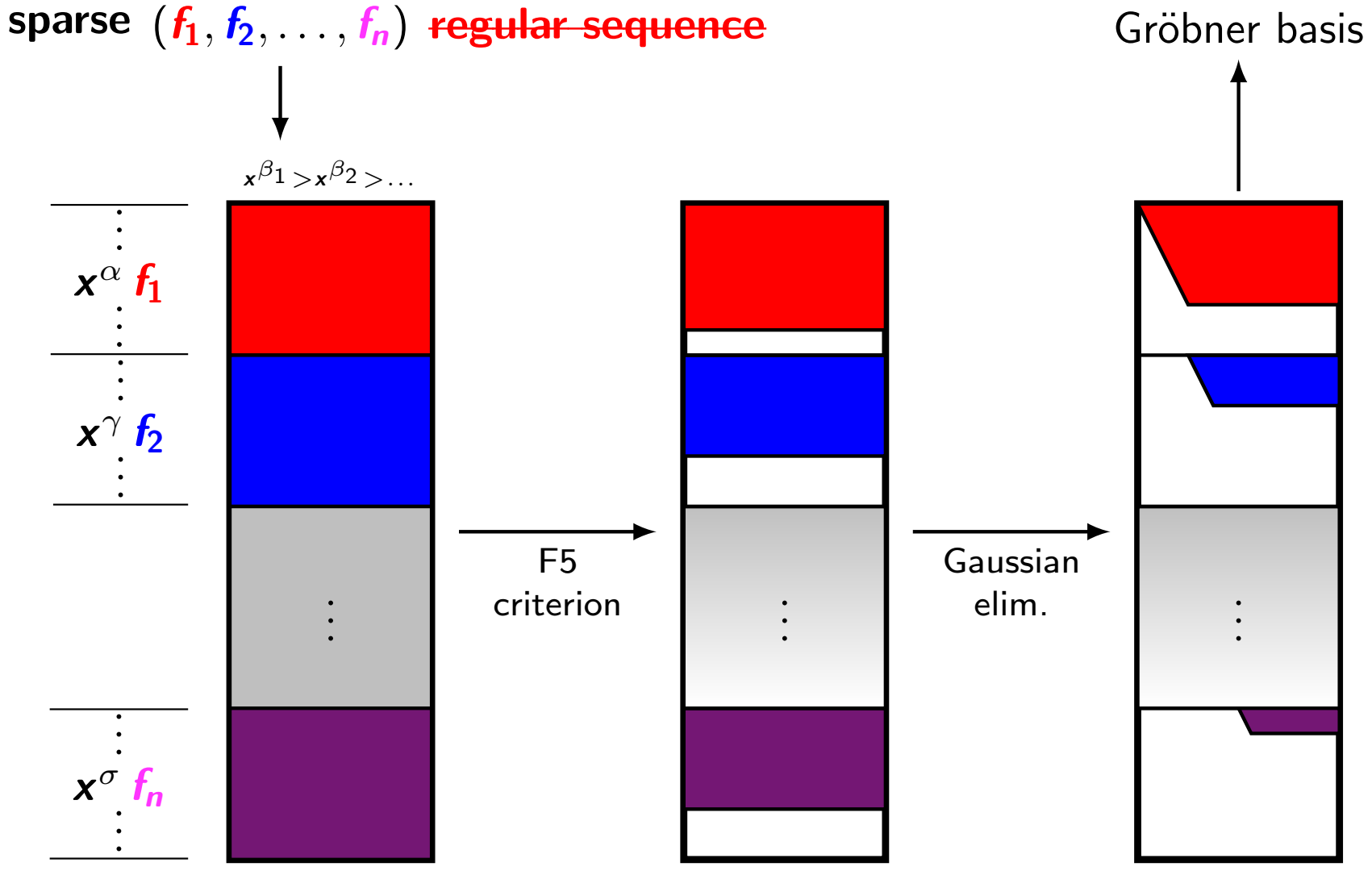

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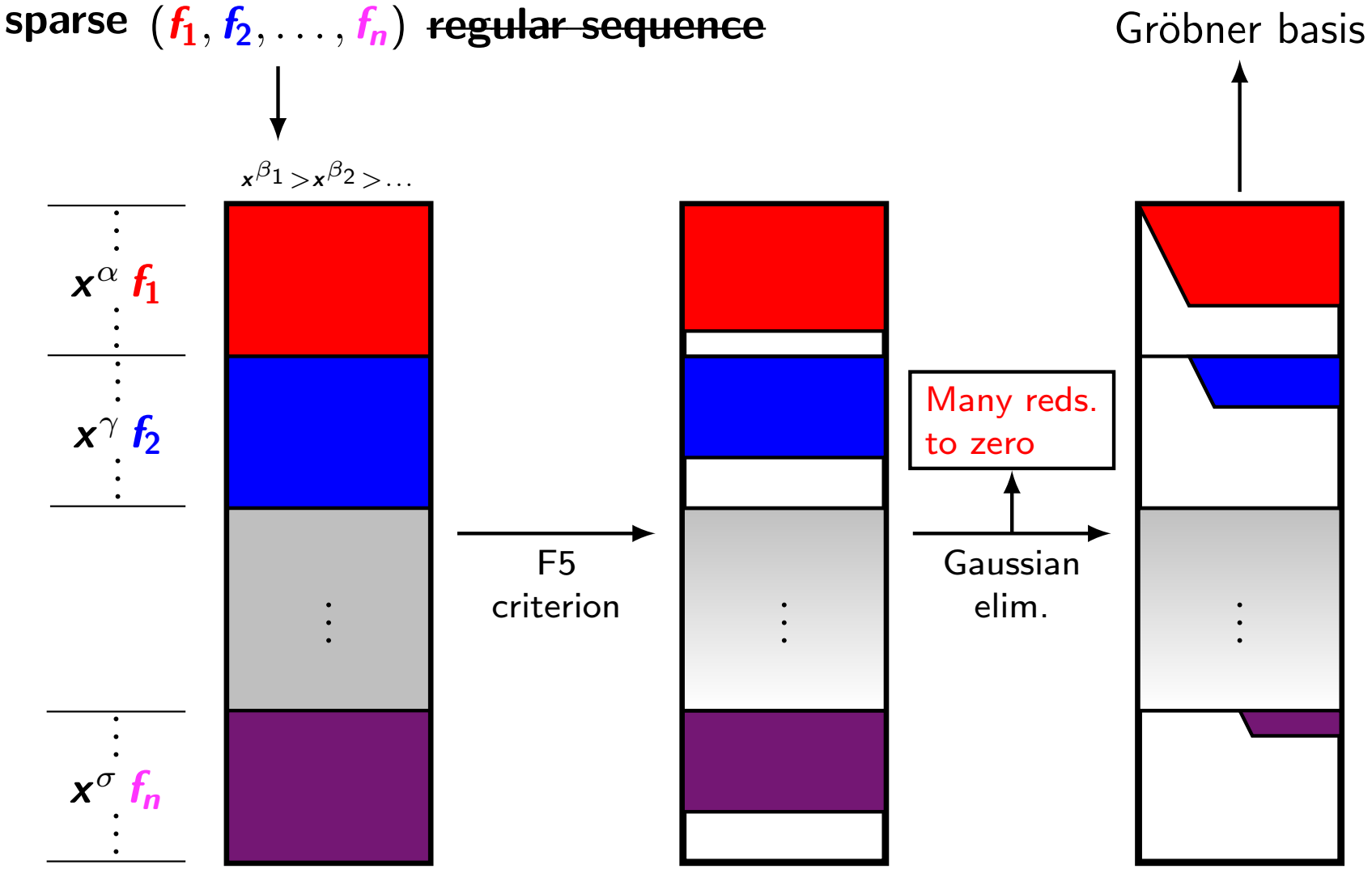

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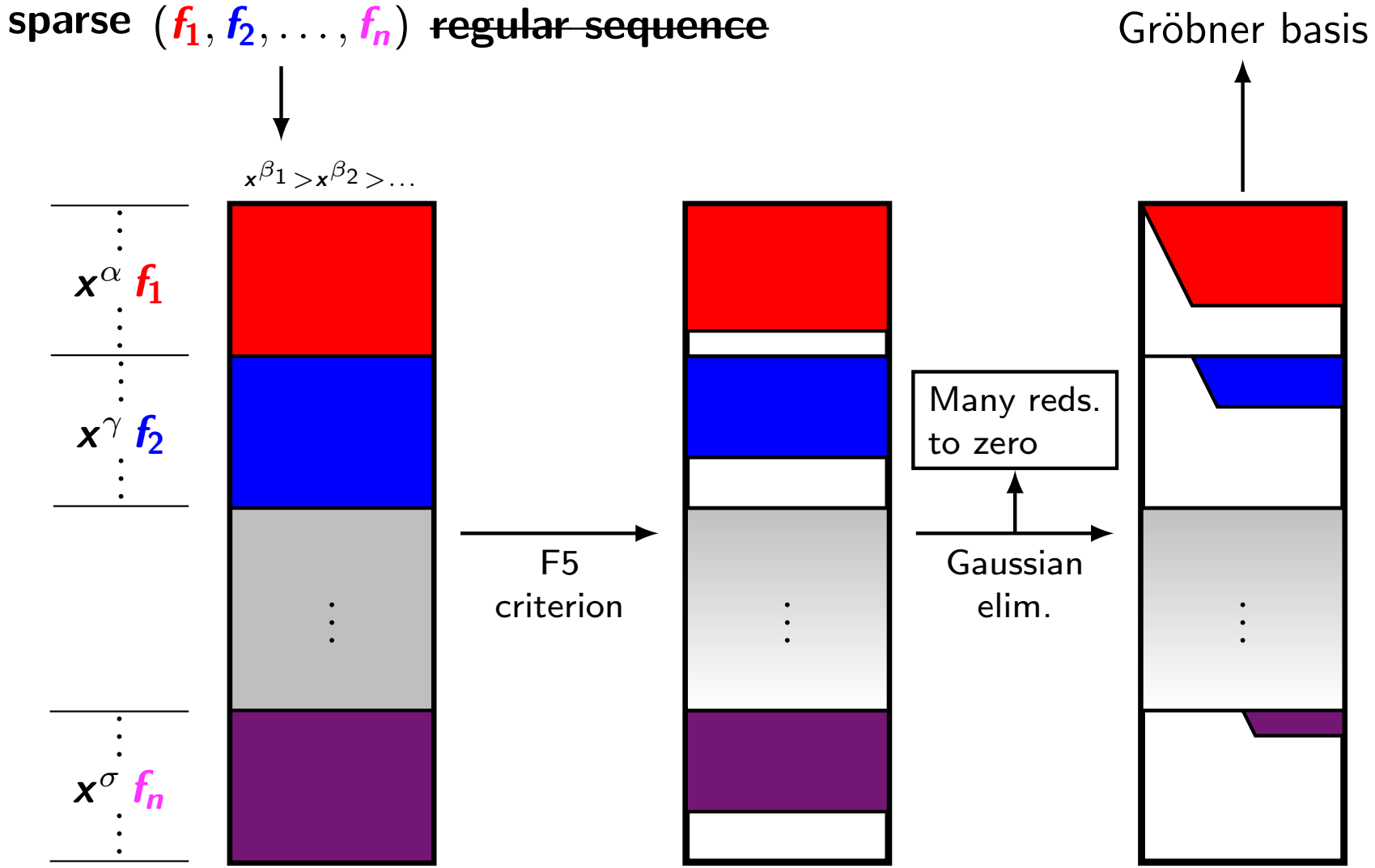

Complexity $\leftrightarrow$ Size of matrix $\rightarrow$ No Macaulay bound for d

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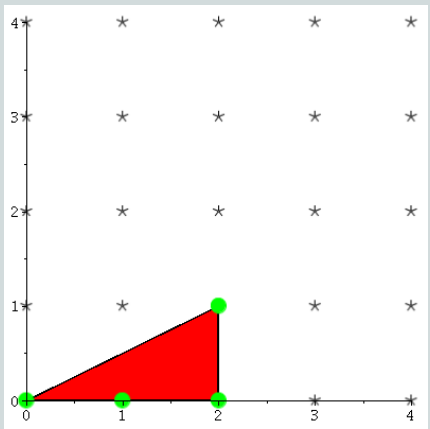
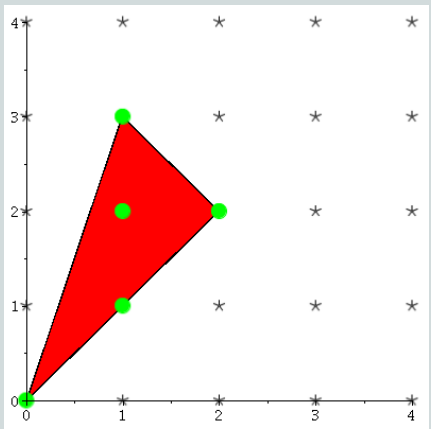


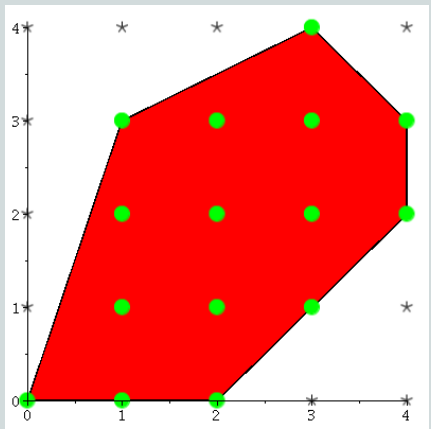
P_1

$+$



P_2

$=$



$P_1 + P_2$

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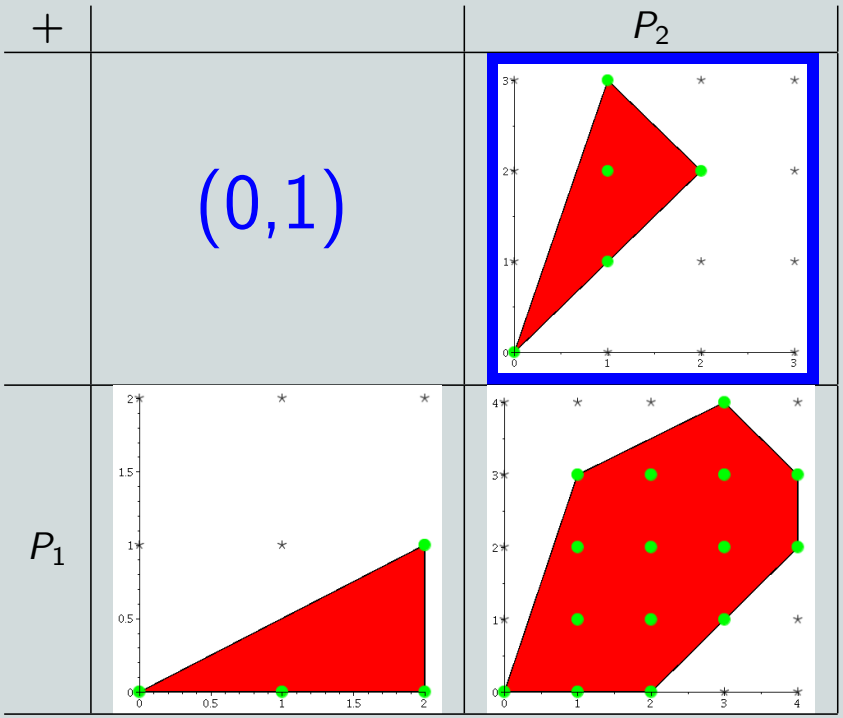

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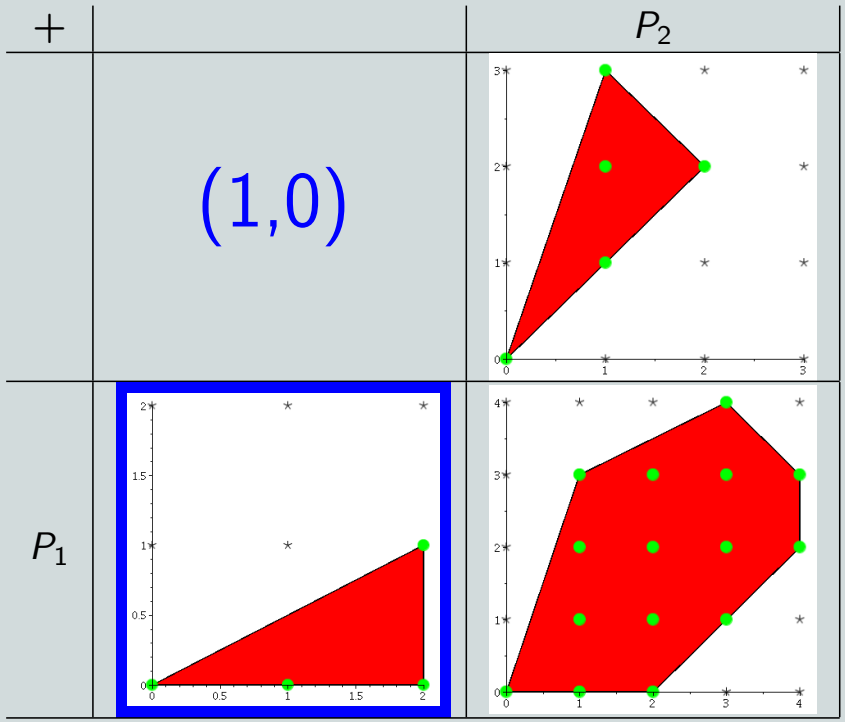

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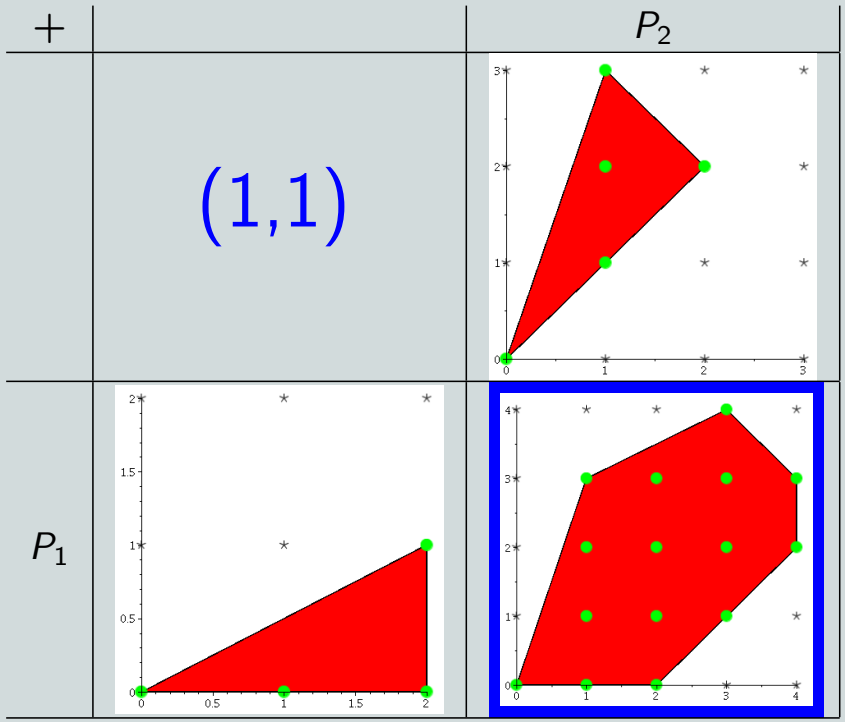

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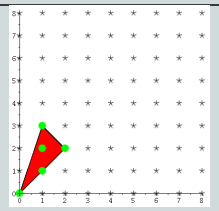
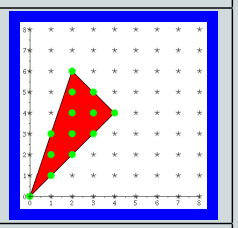
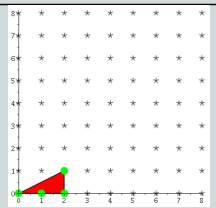
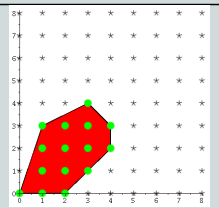
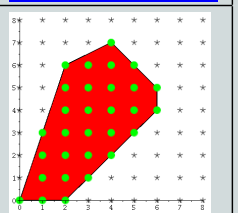
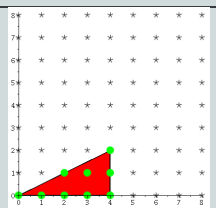
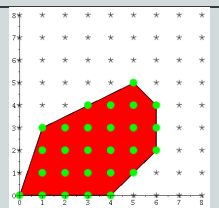
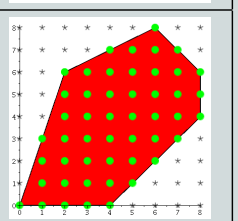
	(0,2)		
P_1			
$2P_1$			

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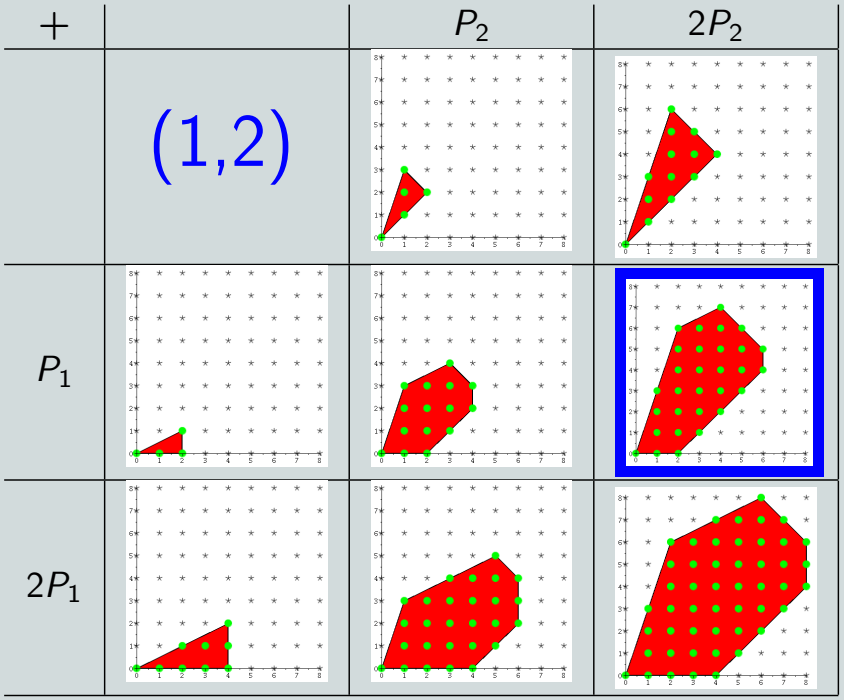

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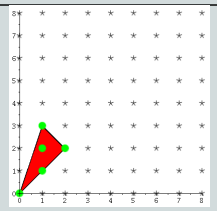
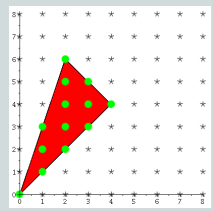
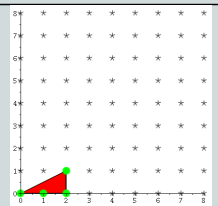
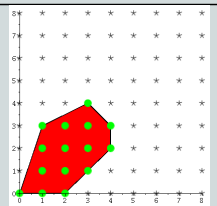
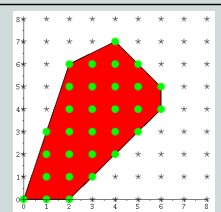
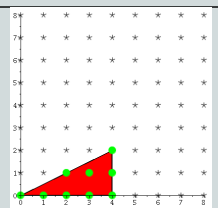
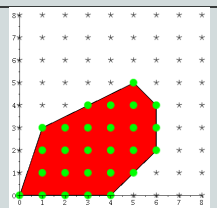
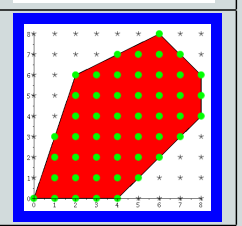
	(2,2)		
P_1			
$2P_1$			

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Complexity for *variant* of [Faugère, Spaenlehauer, Svartz '14].

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Solving numerically sparse systems

Joint work with Simon Telen.

Symbolic-numerical approaches to solve sparse systems

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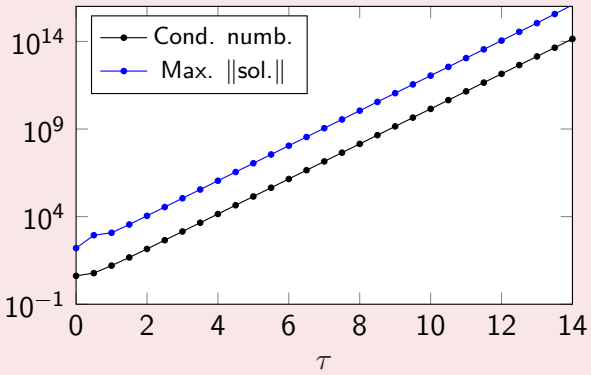
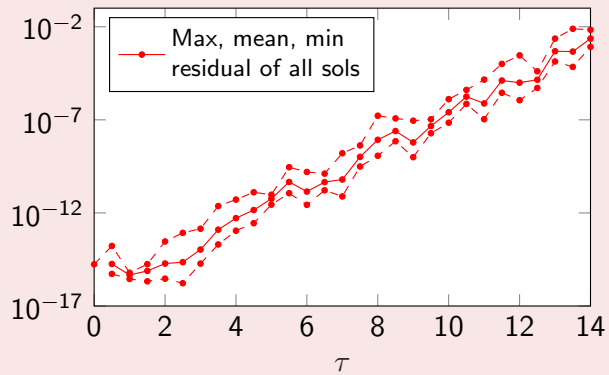
In degenerate situations (e.g. 'sols. going to ∞') $\rightarrow$ Sparse GBs or resultants $\rightarrow$ large numerical errors (for every solution!).

Example

$$\begin{cases} f_1 := -1 + x + x^2 + y + x\,y, \\ f_2 := -1 + x + (\frac{5}{2} - 10^{-\tau})\,x^2 + 2\,y + \frac{5}{2}\,x\,y. \end{cases}$$





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In applications $\rightarrow$ sparse polynomials with noisy coefficients, and we need to approximate solutions.

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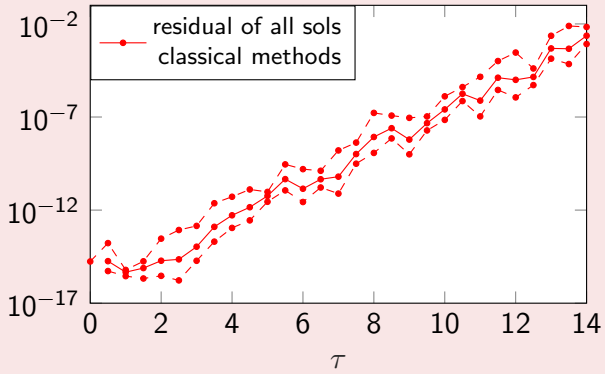

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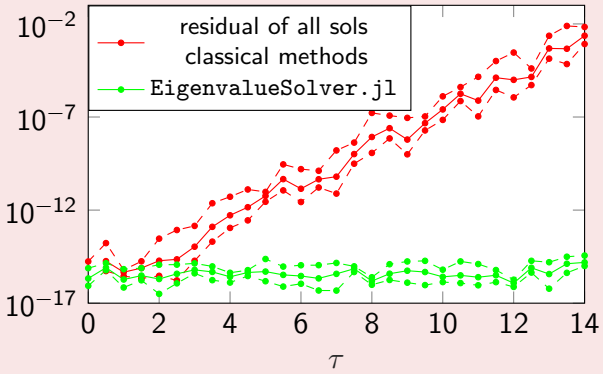

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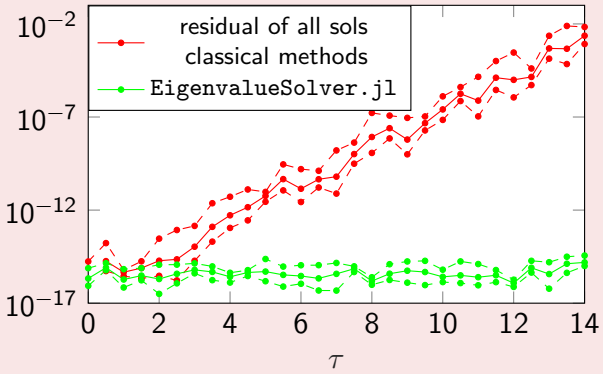

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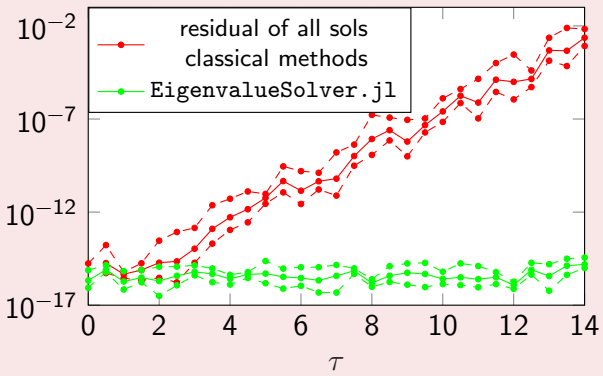

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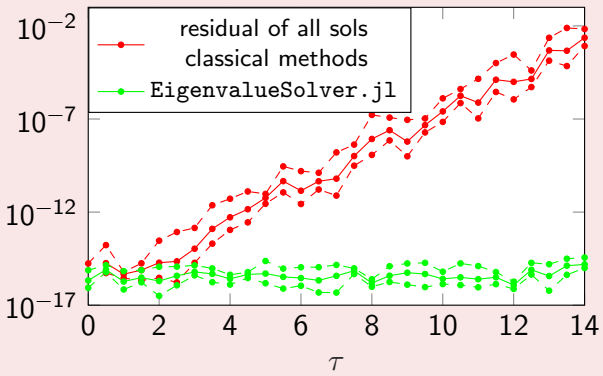

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- Toy implem. in Julia: `EigenvalueSolver.jl`

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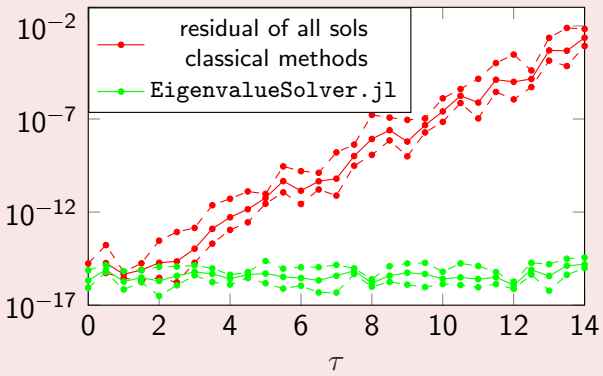

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  - We use a broader class of smaller matrices to solve.

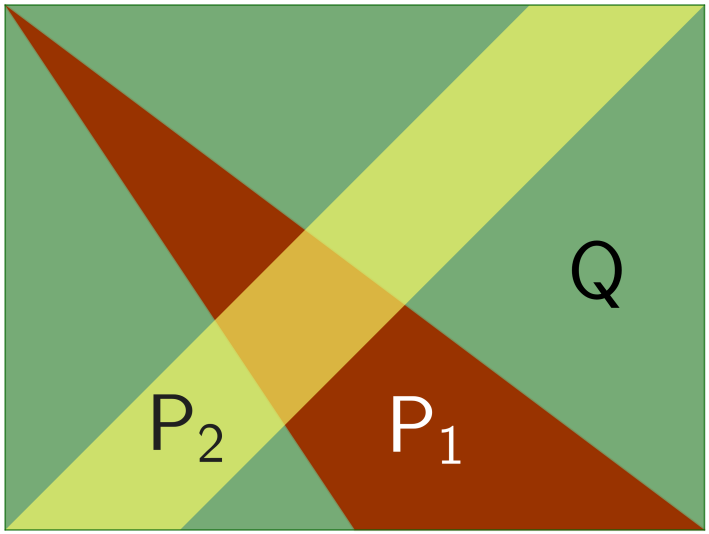
Complete intersections over toric varieties

Joint work with Pierre-Jean Spaenlehauer.

Complete intersections over toric varieties

Given $f_1, \ldots, f_r \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

Assume $P_i \subset Q$.



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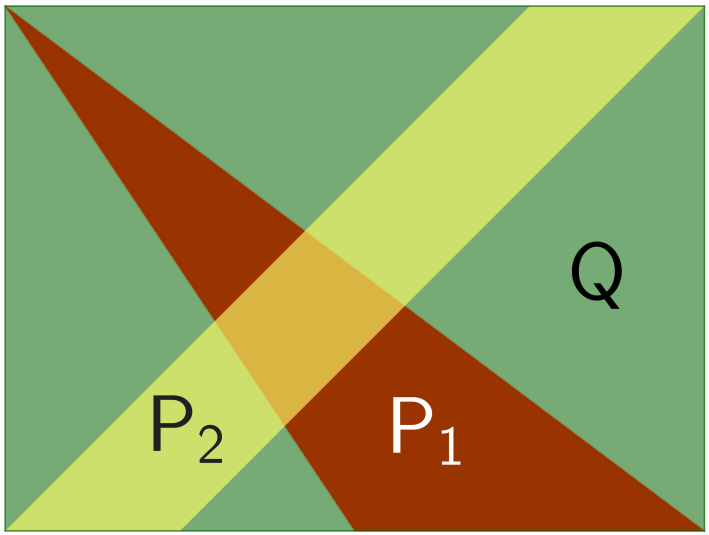

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Unmixed case wrt Q (*)

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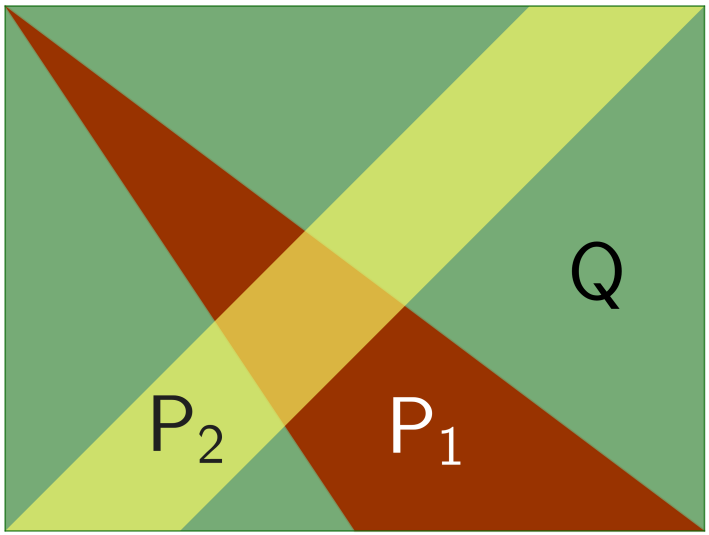


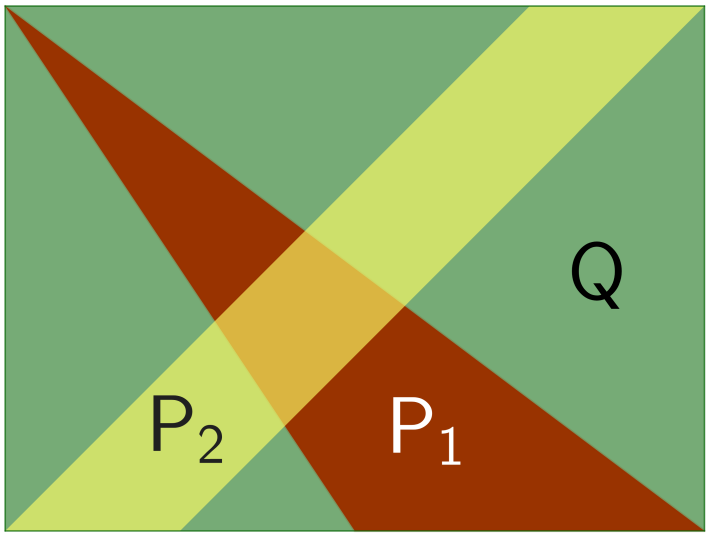
Complete intersections over toric varieties

Given $f_1, \ldots, f_r \in \mathbb{C}[\boldsymbol{x}]$ and **different polytopes** $P_i = \texttt{NewtonPolytope}(f_i)$.

Assume $P_i \subset Q$. The CM regularity bounded by

Mixed case wrt $P_1, \ldots, P_r$ | Unmixed case wrt Q (*)

$$\sum \deg(f_i) \qquad \sum \deg(f_i) - \texttt{codegree}(Q) + 1$$



(*) Requires regularity assumptions!

$\dim(\text{homogenized}(f_1), \ldots, \text{homogenized}(f_i)) = n - i$, for every i.

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In classical projective setting $\mathbb{P}^n$

Fix monomial sets $A_1, \dots, A_r$, s.t. $\deg(x^\alpha) = \deg(x^\beta)$, for every $x^\alpha, x^\beta \in A_i$.
Choose generic system $f_1, \dots, f_r$ such that

$$f_i = \sum_{x^\alpha \in A_i} c_{i,\alpha}\, x^\alpha, \text{ for generic } c_{i,\alpha} \in \mathbb{C}.$$

- We determine $\dim(V_{\mathbb{P}^n}(f_1, \dots, f_r))$.

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Generalizes sufficient cond. ($r = n$) by [Rojas'94],[Bihan&Soprunov'19],[Chen'19].

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Regular sequences for weighted projective spaces [BS22+]

For $\mathbb{C}[\boldsymbol{x}]$ such that $deg(x_i) = w_i$. A generic system with degrees $d_1, \dots, d_r$ is a regular sequence, if and only if, for all $S \subset \{0, \dots, n\}$,

$$|\{i : d_i \in \sum_{j \in S} w_j \, \mathbb{Z}_{\geq 0}\}| \geq |S| + r - n - 1.$$

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Necessary for specialized GB algorithms [Faugère,Safey El Din&Verron'16]

Determinantal formulas for mixed multilinear systems

Joint work with J.-C. Faugère, A. Mantzaflaris & E. Tsigaridas.

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Generalization of solving strategy for Sylvester maps [**B**FMT18]

- Solve using general class of resultant formulas.
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- Solve by computing eigenvalues. (Matrices are not mult. maps!)

Example : Solving using Koszul-type formula

$$\left\{\begin{array}{l} f_1 := 7\,x_0y_0 + -8\,x_0y_1 + -1\,x_1y_0 + 2\,x_1y_1 \\ f_2 := -5\,x_0y_0 + 7\,x_0y_1 + -1\,x_1y_0 + -1\,x_1y_1 \\ f_3 := -6\,x_0z_0 + 9\,x_0z_1 + -1\,x_1z_0 + -2\,x_1z_1 \end{array}\right\} \begin{array}{l} \in \mathbb{C}[\boldsymbol{X}, \boldsymbol{Y}] \\ \in \mathbb{C}[\boldsymbol{X}, \boldsymbol{Z}] \end{array}$$

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$$M = \begin{bmatrix} & & & 5 & -7 & 1 & 1 & & & \\ & & & 7 & -8 & -1 & 2 & & & \\ & -1 & & & & & & -1 & -5 & 7 \\ 7 & & -1 & & & & & -1 & & -5 \\ & 1 & & & & & & -2 & -7 & 8 \\ 8 & & -2 & & & & & 1 & & -7 \\ & 2 & & 9 & & -2 & & -2 & -1 & 2 \\ 2 & & -2 & & 9 & & -2 & 2 & & -1 \\ & 1 & & -6 & & -1 & & 2 & 3 & -4 \\ -4 & & 2 & & -6 & & -1 & 1 & & 3 \end{bmatrix}$$

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(f_1, f_2, f_3) has 2 solutions

$$\left.\begin{array}{l} (1{:}1\ ;\ 1{:}1\ ;\ 1{:}1) \\ (1{:}3\ ;\ 1{:}2\ ;\ 1{:}3) \end{array}\right\} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

Eigenvalues of $\widetilde{M}_{2,2}$

$$\frac{f_0}{x_0y_0z_0}\left((1{:}1\ ;\ 1{:}1\ ;\ 1{:}1)\right) = 3$$
$$\frac{f_0}{x_0y_0z_0}\left((1{:}3\ ;\ 1{:}2\ ;\ 1{:}3)\right) = 1$$

Determinantal formulas for mixed multilinear systems

- We can solve using matrix $M(\boldsymbol{f})$ (in certain class) [BFMT18]

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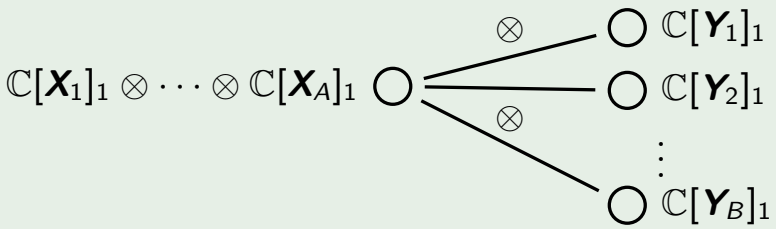
Star multilinear system:

$$f_k \in \mathbb{C}[\boldsymbol{X}_1]_1 \otimes \dots \mathbb{C}[\boldsymbol{X}_A]_1 \otimes \mathbb{C}[\boldsymbol{Y}_{j_k}]_1.$$



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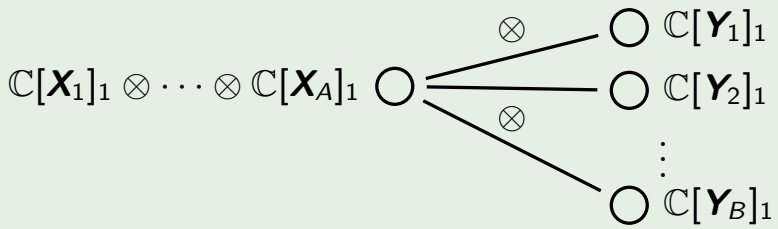
Star multilinear system:

$$f_k \in \mathbb{C}[\boldsymbol{X}_1]_1 \otimes \dots \mathbb{C}[\boldsymbol{X}_A]_1 \otimes \mathbb{C}[\boldsymbol{Y}_{j_k}]_1.$$



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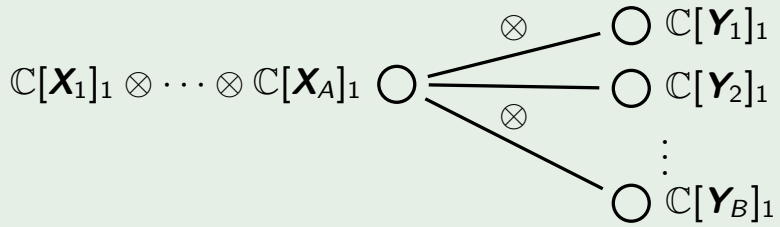

Bipartite bilinear system:

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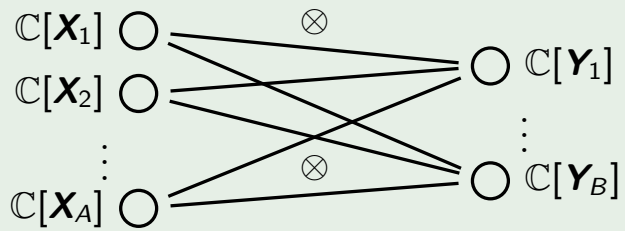

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Generalized Eigenvalue Problem

$$\left(\begin{bmatrix} -7 & -3 \\ -8 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 12 & 2 \\ 13 & 1 \end{bmatrix} \right) \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0 \quad .$$

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$$\left(\lambda_0 \begin{bmatrix} -11 & -3 \\ 4 & 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 7 & -1 \\ 1 & 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 & 0 \\ -1 & -1 \end{bmatrix}\right) \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = 0 \,.$$

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Binary form decomposition

Joint work with J.-C. Faugère, L. Perret & E. Tsigaridas.

Binary form decomposition

Binary form of degree $D \longleftrightarrow$ Symmetric tensor of dimension $2 \times \cdots \times 2$ and order D.

Problem

Given a binary form f of degree D, $a_i \in \mathbb{K} \subset \mathbb{C}$,

$$f(x,y) := \sum_{i=0}^{D} a_i \, x^i y^{D-i} \in \mathbb{C}[x,y]_D$$

compute $\lambda_1, \ldots, \lambda_r$, $\alpha_1, \ldots, \alpha_r$, $\beta_1, \ldots, \beta_r \in \bar{\mathbb{K}}$ and r (**rank**) minimal such that,

$$f(x,y) = \sum_{j=1}^{r} \lambda_j \, (\alpha_j \, x + \beta_j \, y)^D$$

Results

[BFPT16],[BFPT21]

(ACM's SIGSAM Distinguished Student Author Award at ISSAC 2016)

- Quasi-optimal algorithm to decompose binary forms in $\widetilde{O}(D)$ ops.
- Tight bounds for the algebraic degree of the decomposition.

Other applications

Topological data analysis [B., Gäfvert, Lesnick '22+]

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Computational biology [Schwieger, B., Siebert & Haase '20]

- Gröbner-basis-based approach to construct classifiers for Boolean networks.