Sampling relatively factorizable elements in ideals of number fields

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Outline of the talk

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1. Definitions and problem statement
2. Our result
3. Application: computing units

Number fields

- K number field
- R its ring of integers
- n its degree
  - $n = 512$
  - $K = \mathbb{Q}[X]/(X^n + 1)$
  - $R = \mathbb{Z}[X]/(X^n + 1)$

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- n its degree
  - $n = 512$
  - $K = \mathbb{Q}[X]/(X^n + 1)$
  - $R = \mathbb{Z}[X]/(X^n + 1)$
- n measures the "bit-size" of K
  (assume $\log |\Delta_K| \approx n$ for the talk)
  - efficient algorithm $\Leftrightarrow$ poly(n)

Ideals and units

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  - e.g. $\langle 2 \rangle = \{\text{even numbers}\}$; $\langle 1/2 \rangle = \{r/2, r \in \mathbb{Z}\}$
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Representation

An ideal $I = \langle g \rangle$ is represented by a basis $x_1, \cdots, x_n \in I$ such that

$$I = \{\sum_i n_i \cdot x_i \, , \, n_i \in \mathbb{Z}\}.$$

(I is **not** represented by g)

Properties of ideals

Properties:

- $I \cdot J$ $(\langle g \rangle \cdot \langle h \rangle = \langle gh \rangle)$, I^{-1} $(\langle g \rangle^{-1} = \langle g^{-1} \rangle)$, $\langle 1 \rangle = R$

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- unique factorization: $I = \prod_i \mathfrak{p}_i^{\alpha_i}$, $\alpha_i \in \mathbb{Z}$

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Algebraic norm:

- $\mathcal{N}(I) \in \mathbb{R}_+$ measures the "size" of I ($\approx$ absolute value)
- $\mathcal{N}(IJ) = \mathcal{N}(I) \cdot \mathcal{N}(J)$, $\mathcal{N}(I^{-1}) = \mathcal{N}(I)^{-1}$, $\mathcal{N}(R) = 1$
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Computational problems:

- $I \cdot J$, I^{-1}, $\gcd(I, J)$, $\mathcal{N}(I) \Rightarrow \mathrm{poly}(n)$
- recovering a generator g of $I \Rightarrow \mathrm{poly}(n)$ quantum or $L_{2/3}(n)$ classical
- computing the units $R^\times \Rightarrow \mathrm{poly}(n)$ quantum or $L_{2/3}(n)$ classical

Problem statement

Let $\mathcal{S} \subseteq \{I \text{ ideal}\}$. E.g.,

- B-smooth ideals: $\mathcal{S} = \{I = \prod_i \mathfrak{p}_i^{\alpha_i} \,|\, \mathcal{N}(\mathfrak{p}_i) \leq B\}$
- near-prime ideals: $\mathcal{S} = \{I = \mathfrak{p}_0 \cdot \prod_{i \geq 1} \mathfrak{p}_i^{\alpha_i} \,|\, \mathfrak{p}_0 \text{ prime}, \mathcal{N}(\mathfrak{p}_i) \leq B\}$
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  - [RSW18], proven

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[BF14] Biasse and Fieker. Subexponential class group and unit group computation in large degree number Fields. LMS Journal of Computation and Mathematics.

[BP17] de Boer and Pagano. Calculating the power residue symbol and ibeta. ISSAC.

[RSW18] Rosca, Stehlé and Wallet. On the Ring-LWE and Polynomial-LWE problems. Eurocrypt.

Our result

Recall: Problem ★

Given an ideal $I = \langle g \rangle$, find $x = gr \in I$ such that $\langle x \rangle \cdot I^{-1} = \langle r \rangle \in \mathcal{S}$.

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For some sets $\mathcal{S}$, we can solve Problem ★ provably (i.e., without heuristics) in $\mathrm{poly}(n)$ time.

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$\mathcal{S}$ should satisfy

- a random ideal I is in $\mathcal{S}$ with non-negligible probability
- for some $B = \mathrm{poly}(n)$, if $I \in \mathcal{S}$, then $I \cdot J \in \mathcal{S}$ for any J that is B-smooth

Outline of the talk

General idea

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Given an ideal $I = \langle g \rangle$, find $x = gr \in I$ such that $\langle x \rangle \cdot I^{-1} = \langle r \rangle \in \mathcal{S}$.

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Heuristic:

$$\Pr_{x \leftarrow I}(x \cdot I^{-1} \in \mathcal{S}) \approx \delta_S[\beta^n],$$

$\delta_S[\beta^n] = \frac{|\mathcal{S} \cap \{I \,|\, \mathcal{N}(I) \leq \beta^n\}|}{|\{I \,|\, \mathcal{N}(I) \leq \beta^n\}|}$ is the density of $\mathcal{S}$ (among ideals of norm $\leq \beta^n$)

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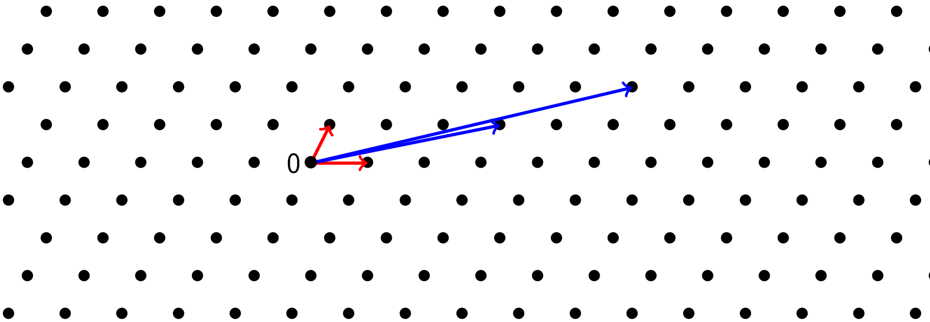
Objective: prove the heuristic

Lattices

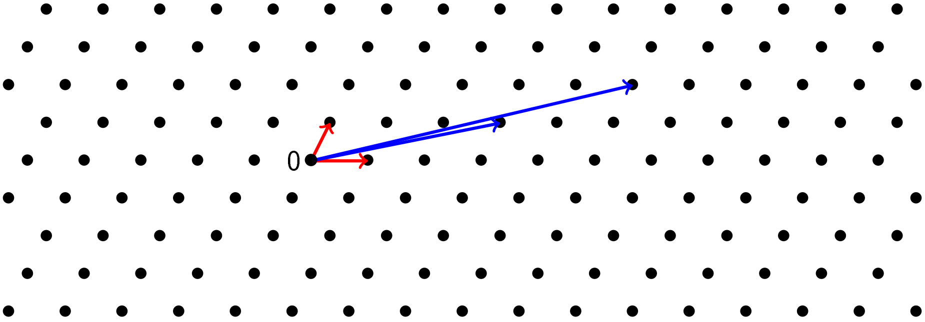


Lattice

A lattice L is a subset of $\mathbb{R}^n$ of the form $L = \{Bx \mid x \in \mathbb{Z}^n\}$, with $B \in \mathbb{R}^{n \times n}$ invertible. B is a basis of L, and n is its rank.

$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 17 & 11 \\ 4 & 2 \end{pmatrix}$ are two bases of the above lattice.

Lattices



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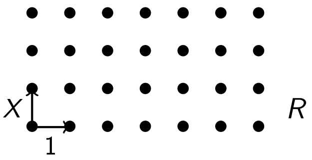
We represent a lattice by any of its basis

Ideals are lattices

$R \simeq \mathbb{Z}^n$

$$R = \mathbb{Z}[X]/(X^n + 1) \to \mathbb{Z}^n$$
$$r = r_0 + r_1 X + \cdots + r_{n-1} X^{n-1} \mapsto (r_0, r_1, \ldots, r_{n-1})$$

(in fact, we actually use Minkowski's embedding)



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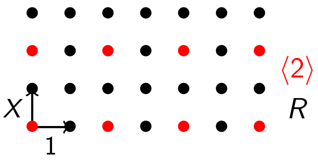
$$\begin{cases} \langle g \rangle \subseteq R \simeq \mathbb{Z}^n \\ \text{stable by '+' and '-'} \end{cases} \Rightarrow \text{lattice}$$



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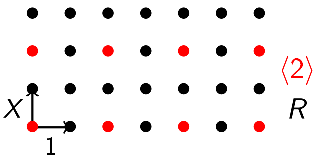
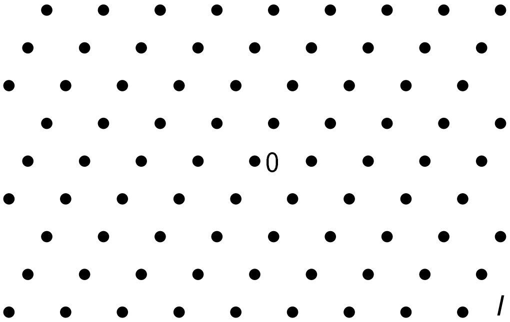


Conclusion: we have an embedding $K \to \mathbb{R}^n$ that maps ideals to lattices

Sampling x in I



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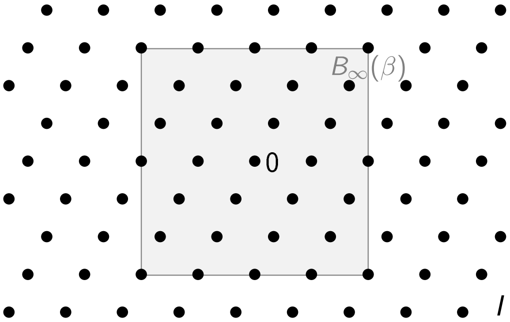

Distribution $\mathcal{D}_I$:

$$x \leftarrow \mathrm{Uniform}(I \cap B_\infty(\beta))$$

(previous works usually used Gaussian distributions)

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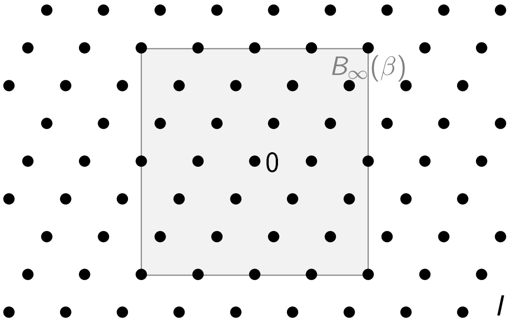

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Efficiency: polynomial time if

$$\beta \geq 2^n$$

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Objective: $p_I := \Pr_{x \leftarrow \mathcal{D}_I}\left(\langle x \rangle \cdot I^{-1} \in \mathcal{S}\right) \approx \delta_{\mathcal{S}}[\beta^n]$

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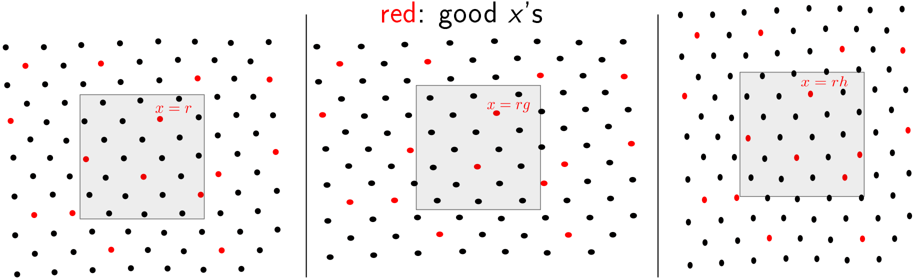
red: good x's



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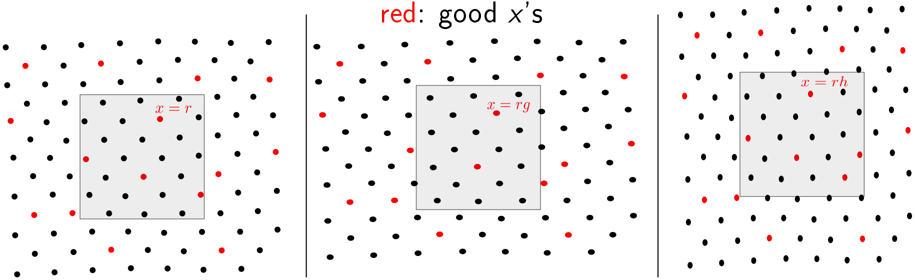

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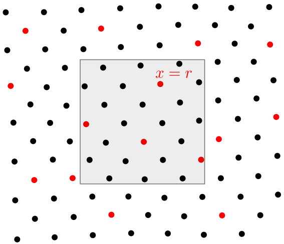
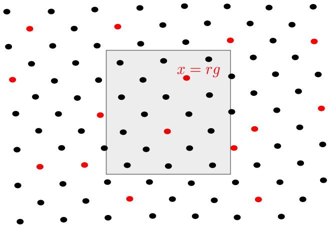


$I = R$

- $\langle x \rangle \cdot I^{-1} = \langle r \rangle$
- $p_I = 4/24 = 1/6$



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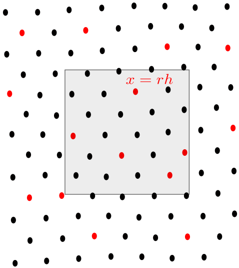

$I = \langle h \rangle$

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- $p_I = 5/23$

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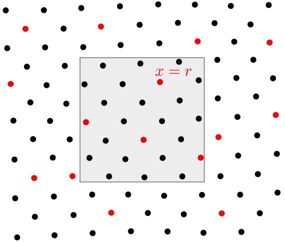

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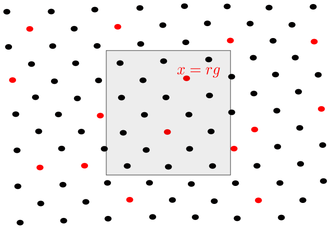

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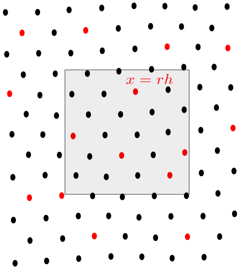

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Idea: Randomize the ideal $I \Rightarrow \Pr_{\substack{I \text{ random} \\ x \leftarrow \mathcal{D}_I}}\left(\langle x \rangle \cdot I^{-1} \in \mathcal{S}\right) \approx \delta_{\mathcal{S}}[\beta^n]$

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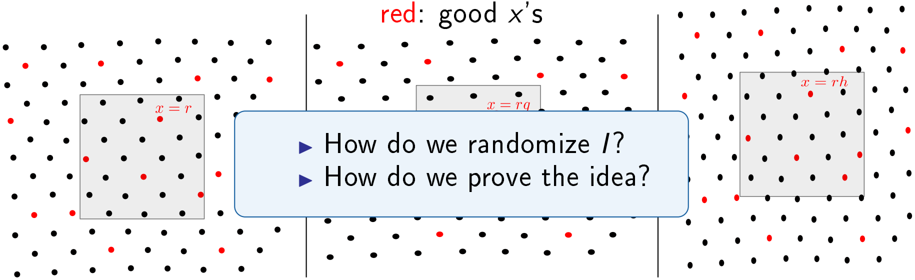

- How do we randomize I?
- How do we prove the idea?

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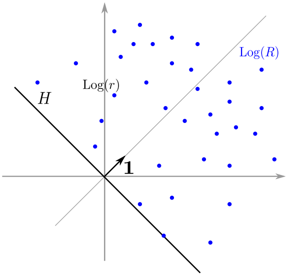

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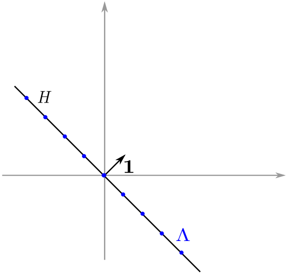
- $\sum_i (\mathrm{Log}(r))_i = 0$ iff r is a unit
- $\Lambda := \mathrm{Log}(R^\times)$ is a lattice



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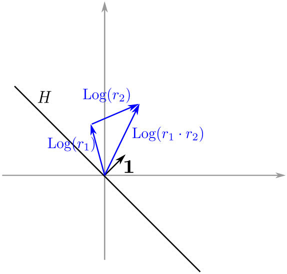

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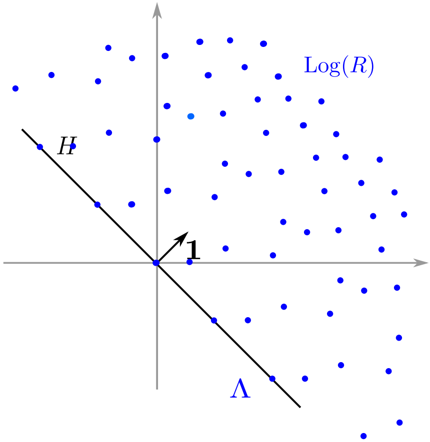

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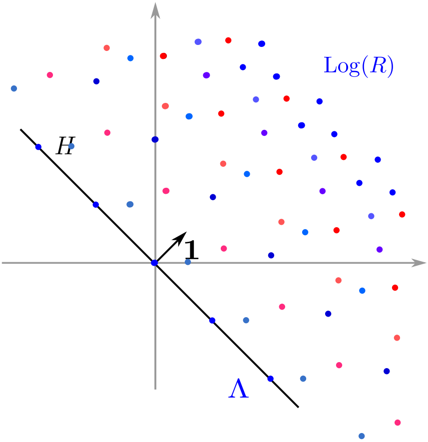

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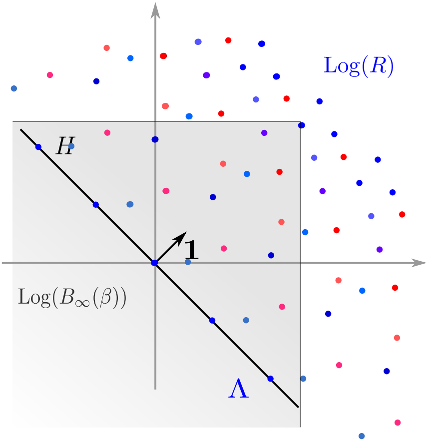

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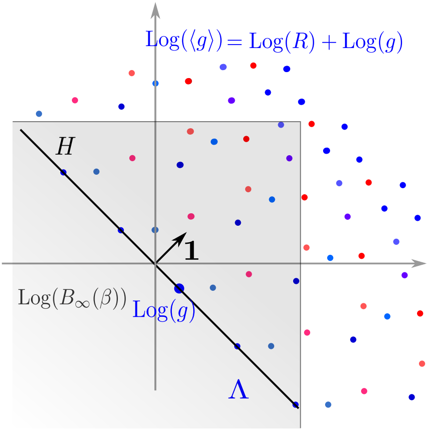
What happens in the Log space



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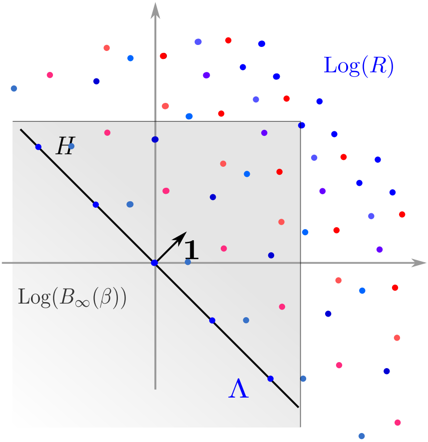

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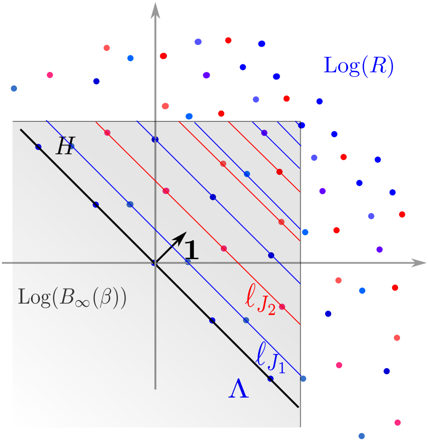

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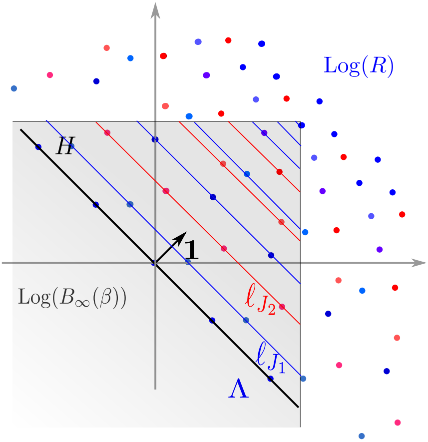

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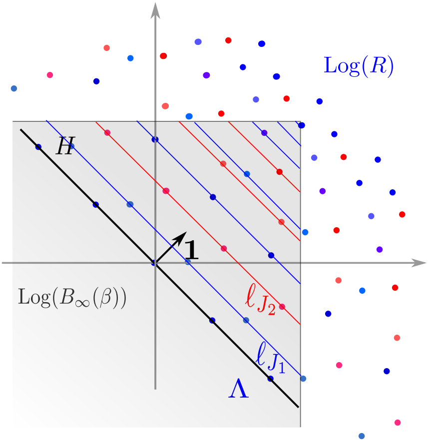

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Randomizing the ideal I

Theorem [BDPW20]

For some $N = \widetilde{O}(n)$ and $B = \mathrm{poly}(n)$, let $\mathcal{P}_B = \{\mathfrak{p} \text{ prime} \,|\, \mathcal{N}(\mathfrak{p}) \leq B\}$.
Let I be any ideal and let $\mathfrak{p}_i \xleftarrow{\$} \mathcal{P}_B$, then

$$J = I \cdot \mathfrak{p}_1 \cdots \mathfrak{p}_N$$

is uniformly random (i.e., $J = \langle g \rangle$, where $g \leftarrow \mathcal{U}(H \bmod \Lambda)$).

(actually, we also need a small distortion on the space)

[BDPW20] de Boer, Ducas, Pellet-Mary and Wesolowski. Random self-reducibility of ideal-SVP via Arakelov random walks. Crypto.

Summary

Algorithm: Given an ideal I

- Randomize the ideal: $J = I \cdot \mathfrak{p}_1 \cdots \mathfrak{p}_N$, with $\mathfrak{p}_i \xleftarrow{\$} \mathcal{P}_B$
- Sample $x \xleftarrow{\$} J \cap B_\infty(2^n)$
- Repeat until $x \cdot J^{-1} \in \mathcal{S}$.

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Our main theorem

Let x be sampled as in the algorithm above (without the rejection step). Let $\mathcal{S}_B$ be the set of B-smooth ideals. Then

$$\Pr_x(x \cdot I^{-1} \in \mathcal{S} \cdot \mathcal{S}_B) \geq \frac{1}{3}\delta_{\mathcal{S}}[2^{n^2}] - 2^{-n}.$$

(Need a small distortion in the algorithm + definition of $\delta_{\mathcal{S}}$ slightly different)

Outline of the talk

Computing units: general idea

Algorithm:

- Fix a bound B and a factor base $\mathcal{P}_B = \{\mathfrak{p} \text{ prime} \,|\, \mathcal{N}(\mathfrak{p}) \leq B\}$

Nice survey: A. Gélin. Class group computations in number fields and applications to cryptology. Thèse de doctorat.

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$$T_{\text{relation}} \approx 3/\delta_{\mathcal{S}_B}[2^{n^2}] \approx \frac{\sharp\{I \mid \mathcal{N}(I) \leq 2^{n^2}\}}{\sharp\{B\text{-smooth ideal } I \mid \mathcal{N}(I) \leq 2^{n^2}\}}$$

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- A way to prove that the relations we create are independent
  (if we want to find all units)

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Questions?