SQISign: Compact Post-Quantum Signatures from Quaternions and Isogenies

Antonin Leroux, joint work with L. De Feo, D. Kohel, C. Petit and B. Wesolowski

DGA, Ecole Polytechnique, Institut Polytechnique de Paris, Inria Saclay

The State of Post-Quantum Cryptography

Six families still in Round 3 NIST post-quantum competition (Finalists + Alternate Candidates):

Lattices	4 encryption	2 signature
Codes	3 encryption	
Multivariate		2 signature
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Signatures maybe?

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Yoo et al. "A post-quantum digital signature scheme based on supersingular isogenies"

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Galbraith, Petit, and Silva "Identification Protocols and Signature Schemes Based on Supersingular Isogeny Problems"

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De Feo and Galbraith "SeaSign: Compact isogeny signatures from class group actions"

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- [BKV19] CSI-FiSh: Based on CSIDH + precomp. $\Rightarrow$ bad scaling,
  similar to SeaSign with improved efficiency and sizes.

Beullens, Kleinjung, and Vercauteren "CSI-FiSh: Efficient isogeny based signatures through class group computations"

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New security assumption.

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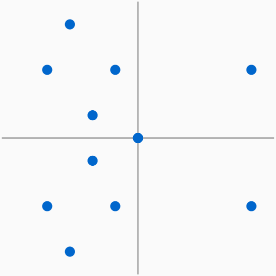
The **degree** is $\deg(\varphi) = \# \ker(\varphi)$.

The **dual** isogeny $\hat{\varphi} : F \to E$

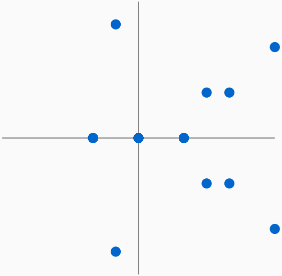
$$\hat{\varphi} \circ \varphi = [\deg(\varphi)]_E$$

Isogenies: an example over $\mathbb{F}_{11}$

$E : y^2 = x^3 + x$



$F : y^2 = x^3 - 4x$



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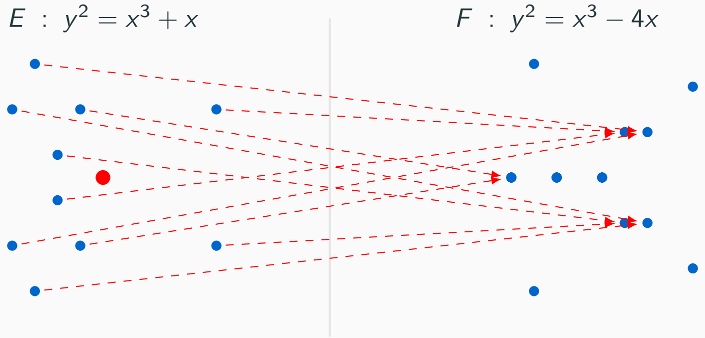

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- Kernel generator in red.
- This is a degree 2 map.
- Analogous to $x \mapsto x^2$ in $\mathbb{F}_q^*$.

Endomorphism ring

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This talk $\to$ supersingular curves.

Key exchange betw. Alice and Bob.

Jao and De Feo "Towards Quantum-Resistant Cryptosystems from Supersingular Elliptic Curve Isogenies"

Supersingular Isogeny Diffie Hellman

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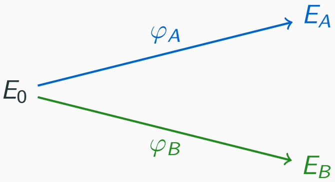

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\begin{tikzcd}
 & E_A \arrow[r, "{[\varphi_A]_*\varphi_B}"] & {} \\
E_0 \arrow[ur, "\varphi_A"] \arrow[dr, "\varphi_B"'] & & \\
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Push-forward kernel $\ker([\varphi]_*\psi) = \varphi(\ker\psi)$.

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Efficient when N_A, N_B are *smooth*.

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Supersingular Isogeny Problem

The underlying *security problem*:

Supersingular ℓ-Isogeny Problem: Given a prime p and two supersingular curves E_1 and E_2 over $\mathbb{F}_{p^2}$, compute an ℓ^e-isogeny $\varphi : E_1 \rightarrow E_2$.

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$$E_1 \xrightarrow{\quad\quad\varphi\quad\quad} E_2$$

The Deuring Correspondence

Quaternion Algebra, Orders and Ideals

The **Quaternion algebra** $H(a, b)$ is

$$H(a, b) = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + k\mathbb{Q} \text{ with } i^2 = a, j^2 = b$$

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Fractional ideals are $\mathbb{Z}$-lattices of rank 4 inside $H(a, b)$

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The (maximal) **left order**[1] $\mathcal{O}_L(I)$ of an *ideal* is

$$\mathcal{O}_L(I) = \{\alpha \in H(a, b), \alpha I \subset I\}$$

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$\mathcal{A}_p$: quaternion algebra depending only on p.

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$\iota : (x, y) \mapsto (-x, \sqrt{-1}y)$ is a twisting automorphism with $\iota \circ \iota = [-1]$.

A new security problem?

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[Koh+14]: *heuristic polynomial* time algorithm `KLPT` for quaternion path problem.

Kohel et al. "On the quaternion ℓ-isogeny path problem"

Algorithmic summary of effective Deuring Correspondence

Problems with ✗ are hard, ✓ are easy. All ✓ are obtained using KLPT in [Eis+18].

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Proof of Knowledge of Endomorphism Ring

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SQISign contributions:

- A new generic `KLPT` algorithm to reach high soundness.
- New algorithmic tools to make the scheme **practical**.

Galbraith, Petit, and Silva "Identification Protocols and Signature Schemes Based on Supersingular Isogeny Problems"

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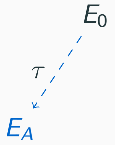

- - - - - secret key isogeny

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$$\begin{tikzcd}
 & E_0 \arrow[dl, dashed, "\tau"'] & \\
E_A \arrow[rr, "\sigma_0"] & & E
\end{tikzcd}$$

$\longrightarrow$ commitment isogeny (prover)

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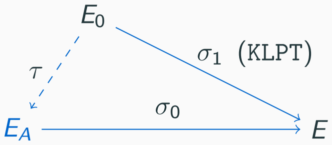

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$$\begin{tikzcd} & E_0 \arrow[dl, dashed, "\tau"'] \arrow[dr, "\sigma_1\ (\texttt{KLPT})"] & \\ E_A \arrow[rr, "\sigma_0"] & & E \end{tikzcd}$$

$b \in \{0, 1\}$

$\longrightarrow$	commitment isogeny (prover)
•	challenge bit (verifier)
- - - - -	secret key isogeny

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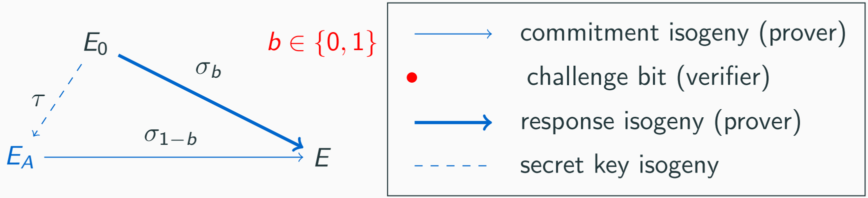

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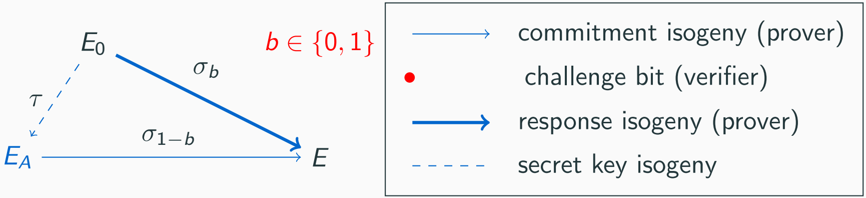

Repeat this λ times to reach 2^{λ}-bits of soundness.

SQISign Identification Scheme

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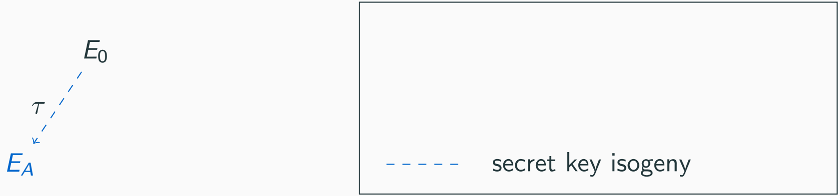

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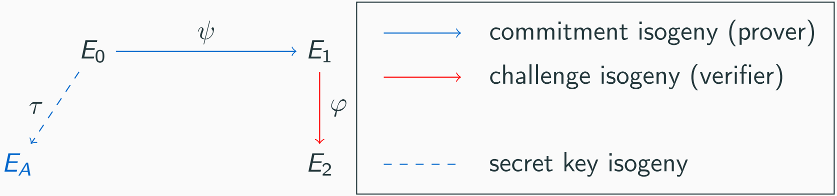

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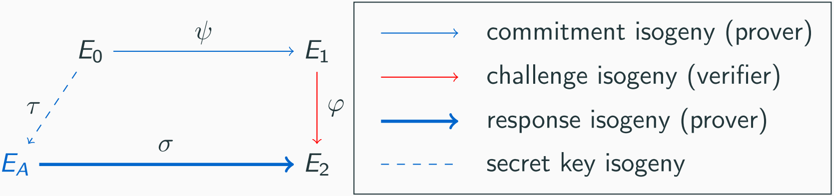

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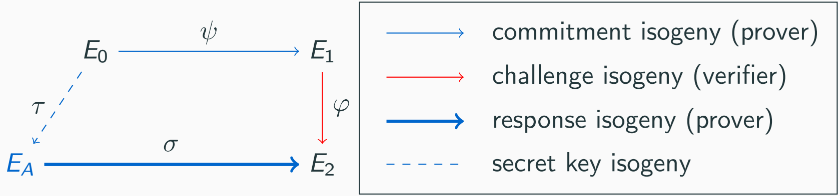

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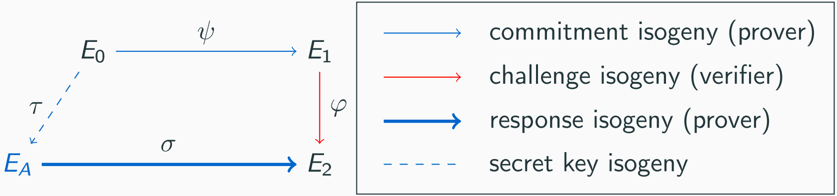

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Zero-Knowledge: prove that σ is a random isogeny $\Rightarrow$ depends on the algorithm computing σ.

SQISign in Practice

Effective Deuring Correspondence: from Ideals to Isogenies

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- Take D powersmooth $\to E[D]$ in $\sim$ small extension ([GPS17]).
- Take $D = \ell^f$ and split σ in smaller isogenies of degree ℓ^e and apply `IdealToIsogeny` for each (**SQISign**).

New Pb: for generic E of known $\mathrm{End}(E)$, hard to evaluate $\mathrm{End}(E)$...

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For fast verification we take σ of degree 2^f, $f = O(\log_2(p))$.

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We found a 256 bits prime p with $e = 33, f = 1000$ and 2^{13}-smooth integer of 395 bits:

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T = \; & 5^{21} \cdot 7^2 \cdot 11 \cdot 31 \cdot 83 \cdot 107 \cdot 137 \cdot 751 \cdot 827 \cdot 3691 \cdot 4019 \cdot 6983 \\
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Bottleneck of the signature: $\Theta(f/e)$ T-isogeny computations .

What now?

Conclusion and Important Problems

We introduced the most compact post-quantum signatures but efficiency is still order of magnitudes below the competitors. The underlying sigma protocol has zero-knowledge property relying on a new assumption.

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- Upgrade the implementation: lots of room for improvement.
- Advance on the `KLPT` algorithm: either for efficiency or security.
- Better understanding of the current ZK assumption.
- Find new algorithms for effective Deuring Correspondence.

Questions?

https://eprint.iacr.org/2020/1240