Verifiable delay functions from elliptic curve cryptography

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Thales – LORIA

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Uniqueness If $\mathrm{Verify}(pp, x, y, \pi) = \mathrm{Verify}(pp, x, y', \pi') =$ yes, then $y = y'$.

Correctness The verification will always succeed if Eval has been computed honestly.

Soundness A lying evaluator will always fail the verification.

Sequentiality It is impossible to correctly evaluate the VDF in time less than $T - o(T)$, even when using $\mathrm{poly}(T)$ parallel processors.

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Idea: slow things down by adding delay.

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- VDF without "no parallelization": pre-image of a hash function.

$$f(x) = h^{-1}(x)$$

  Verification is easy: $h(f(x)) \stackrel{?}{=} x$.
  Computation is faster as long as you parallelize.

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Setup. N is a RSA modulus, public parameters: $(\mathbb{Z}/N\mathbb{Z}, H : \{0,1\}^* \to \mathbb{Z}/N\mathbb{Z})$.

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- If one knows the factorization of N, the evaluation can be computed using

$$H(x)^{2^T} \equiv H(x)^{2^T \bmod \varphi(N)} \mod N$$

Need a *trusted setup* to choose N.

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- It works in class group: Let $K = \mathbb{Q}(\sqrt{-D})$ and O_K its ring of integers.

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- It is not post-quantum...

Let E be an elliptic curve defined over $\mathbb{F}_p$.
Suppose that we have N a large prime integer and k a small integer such that

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$$e(\varphi(P), \varphi(Q)) = e(P, Q)^{\deg(\varphi)}$$

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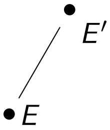

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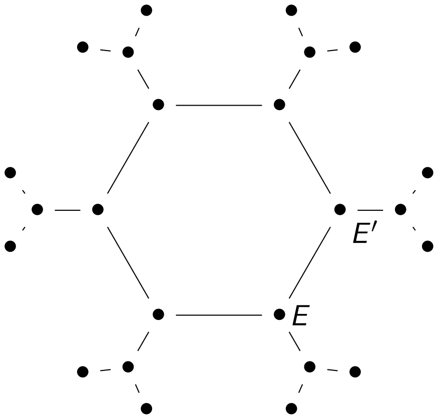
Ordinary curves $End(E)$ is an order in $\mathbb{Q}(\sqrt{-D})$.



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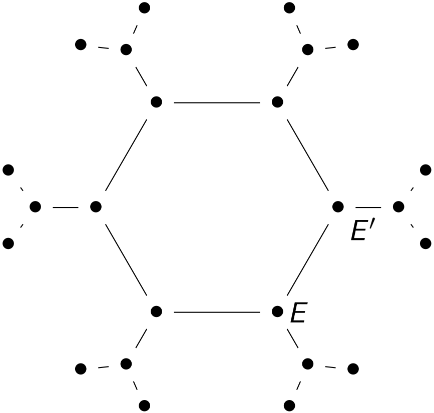

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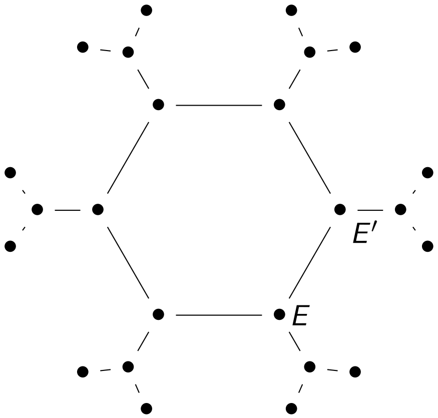

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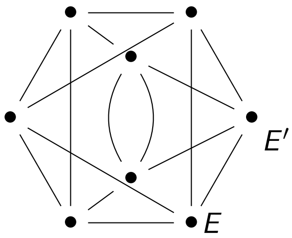

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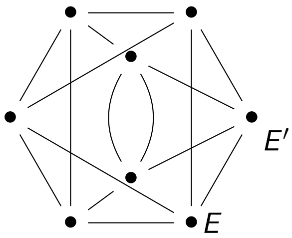
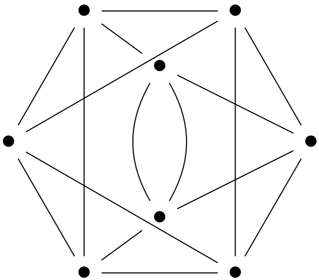


VDF over $\mathbb{F}_{p^2}$ supersingular curves.



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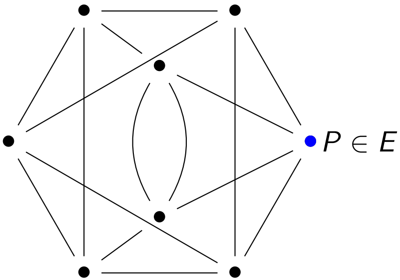

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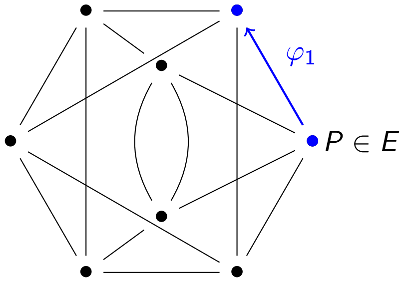

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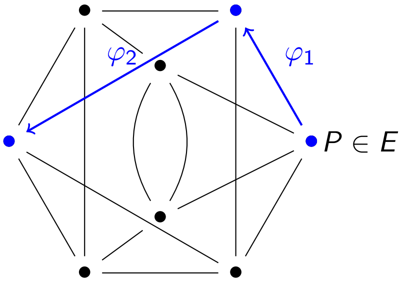

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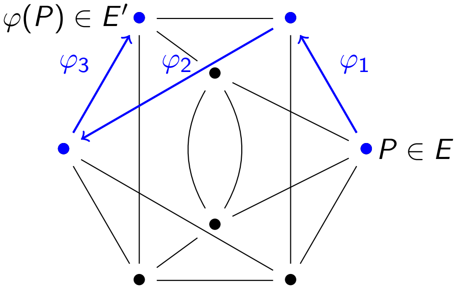

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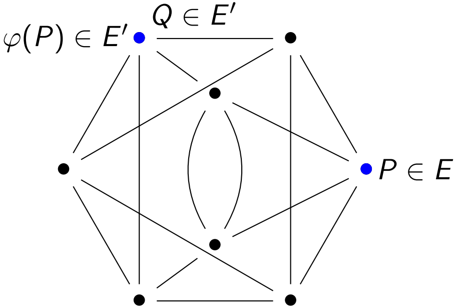

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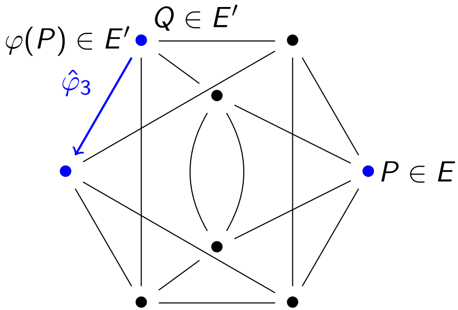

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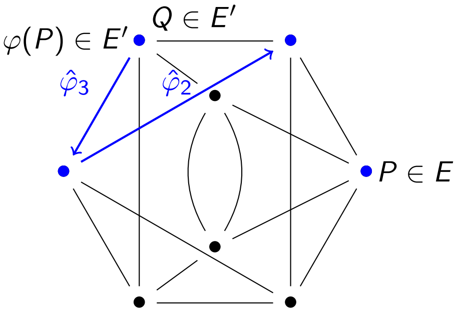

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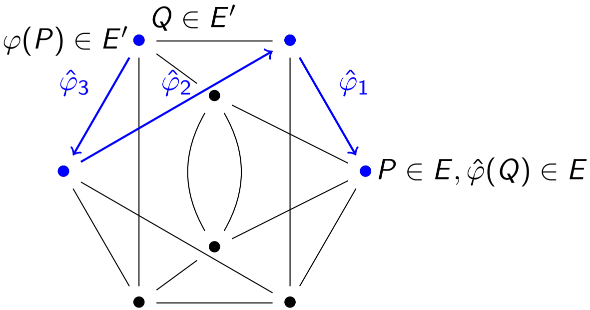

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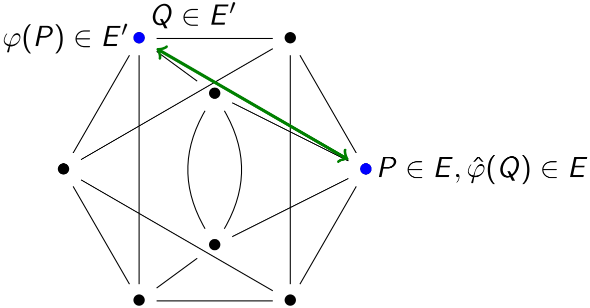

VDF over $\mathbb{F}_{p^2}$ supersingular curves.

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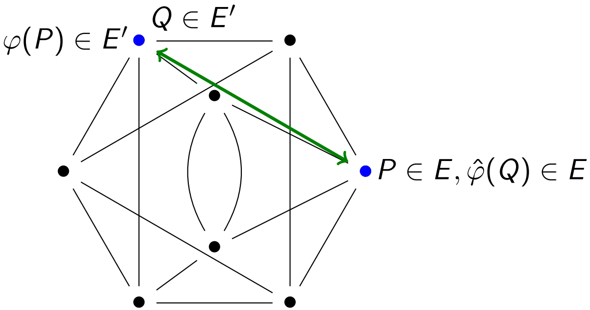

Not post-quantum, but also no proof needed!

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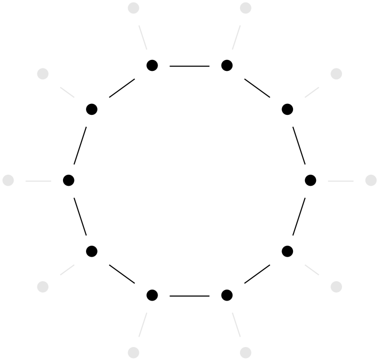

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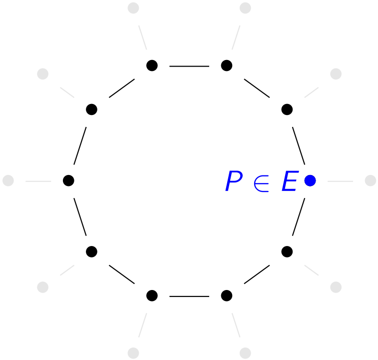

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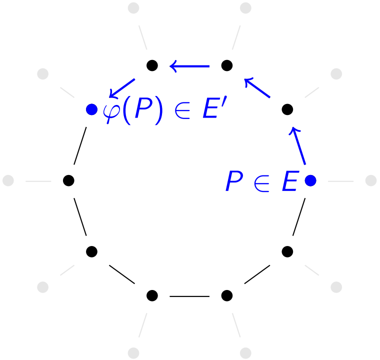

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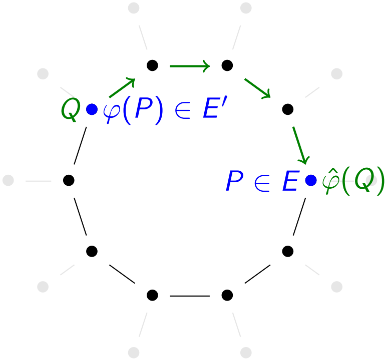

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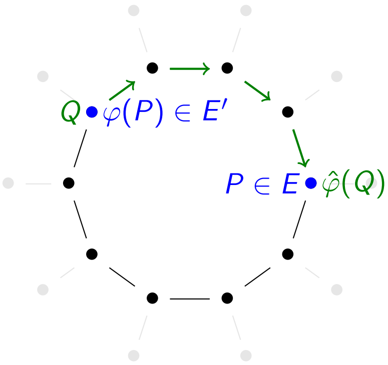

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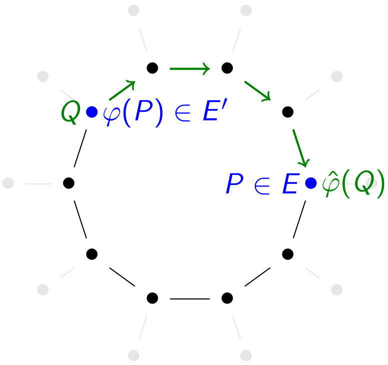

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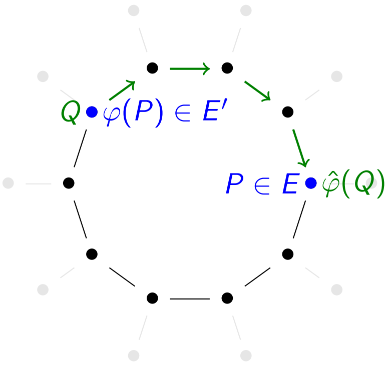

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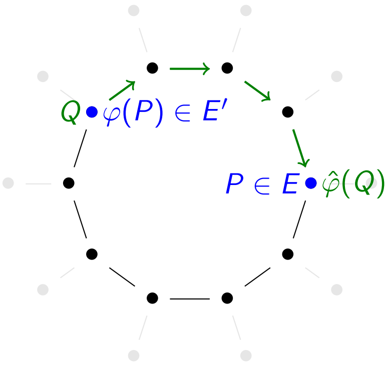

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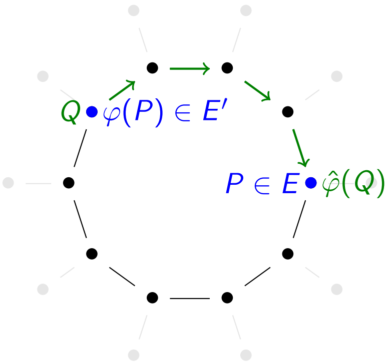

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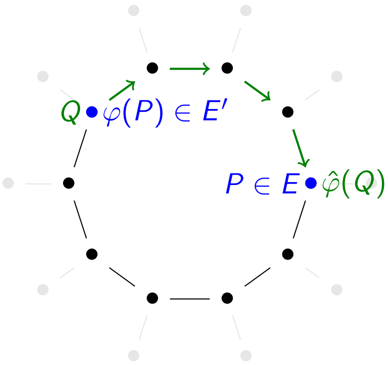

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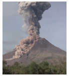

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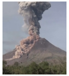

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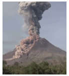

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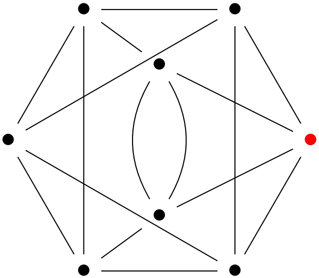

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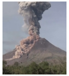

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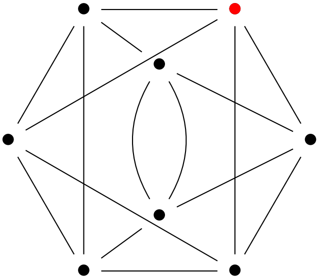

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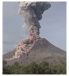

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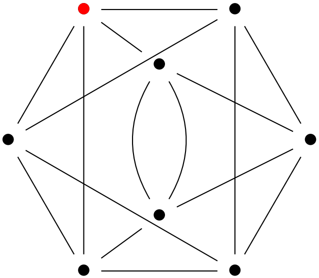

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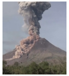

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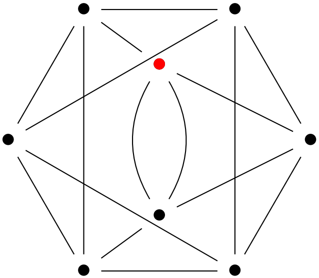

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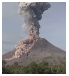

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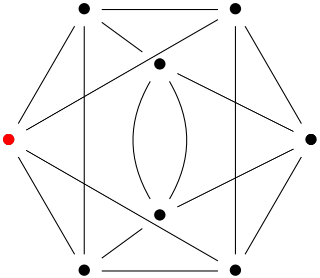

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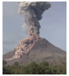

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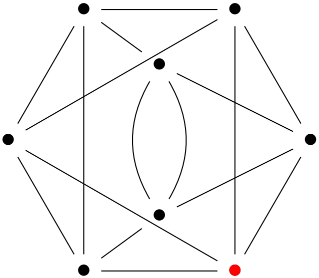

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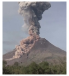

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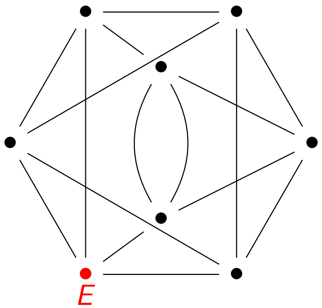

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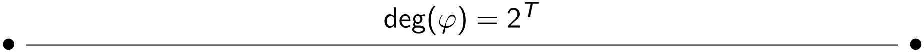

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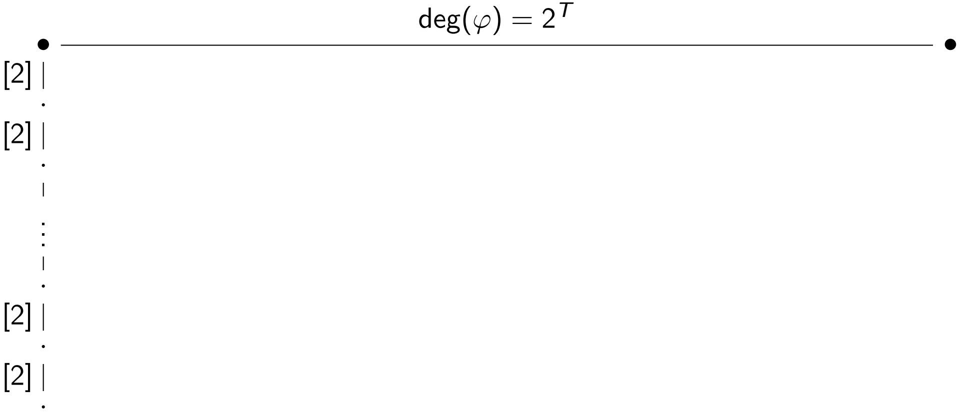

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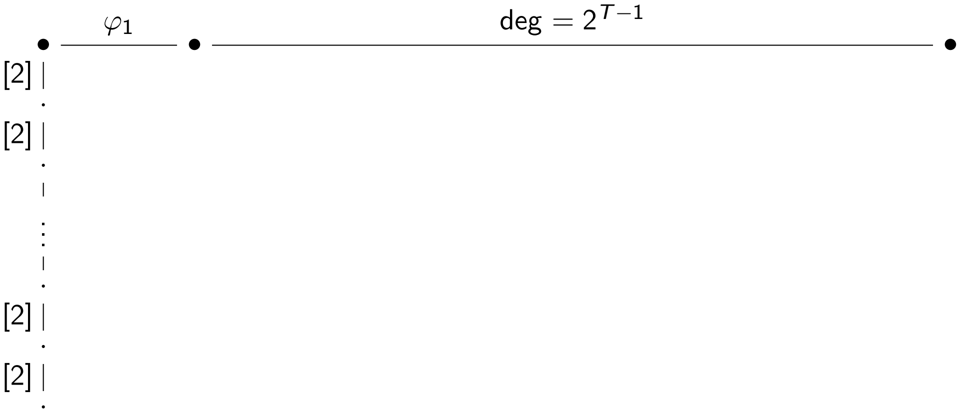

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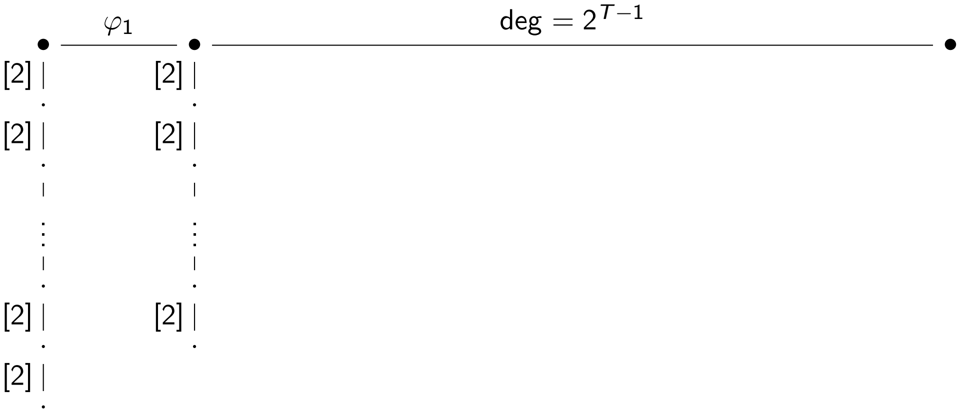

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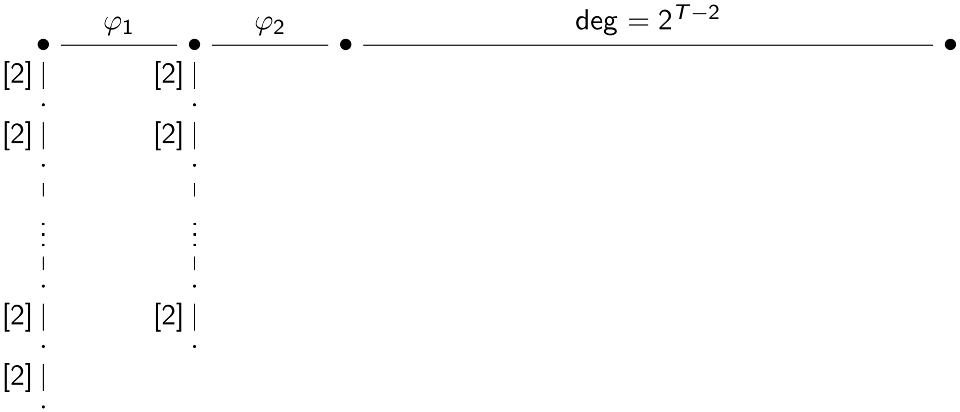

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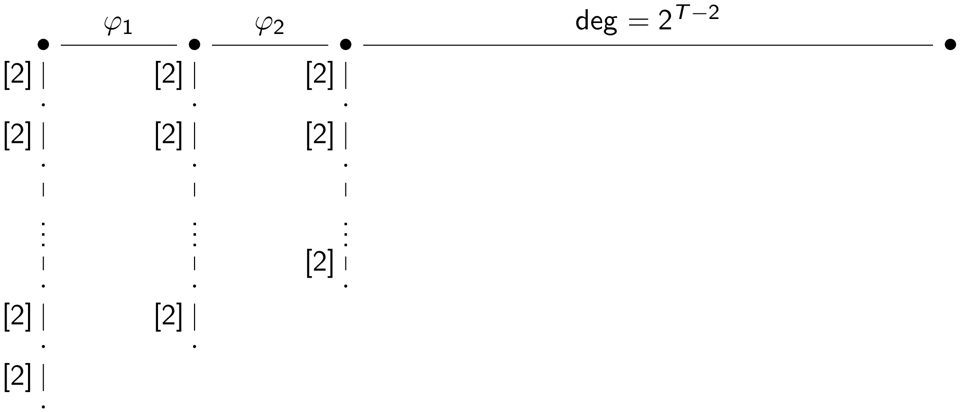

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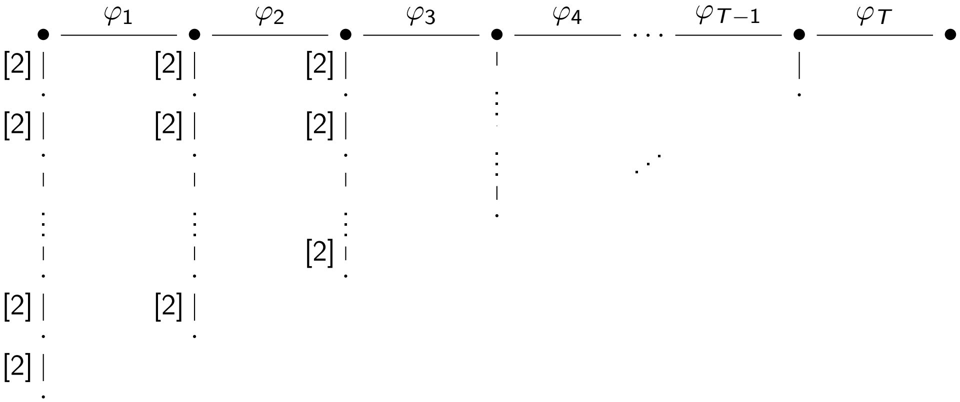

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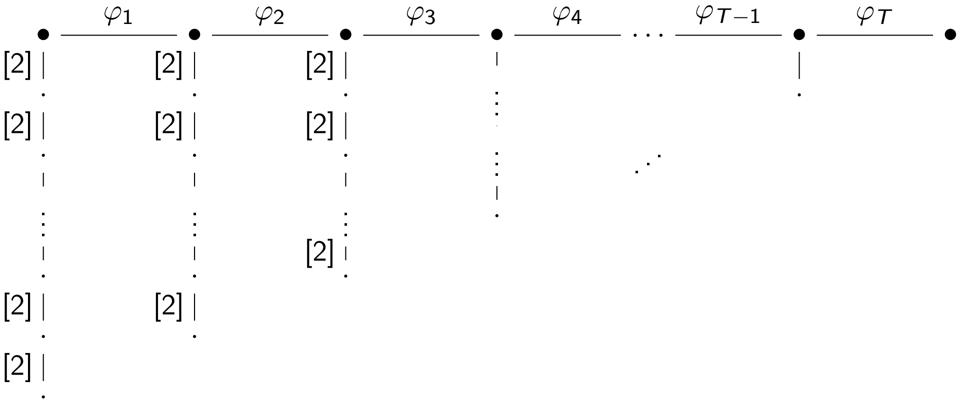

Complexity: $O(T^2)$. It can be turned into $O(T\log_2(T))$ with a recursive strategy.

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$$\#E(\mathbb{F}_p) = p + 1$$

$$p = 2^n f N - 1$$

$$\log_2(p) = 1500$$

Proof-of-concept available on https://github.com/isogeny-vdf.

In practice, we cannot find a point of order 2^T (it is too large).
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- Our VDF over $\mathbb{F}_p$ is **not** quantum-annoying: once the setup is done, a quantum computer can compute the class number $\mathrm{Cl}(-D)$ and then find a faster isogeny (similar to Wesolowski group-class VDF).

A generalization of the BLS signature.

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Let E an elliptic curve and $P \in E(\mathbb{F}_p)$ a point of order N.

- Secret key: s an integer
- Public key: $P_K = \varphi(P)$.

Sign Hash the message m into $\mathbb{G}_2$ and the signature is $\sigma = [s]H(m)$.

Verify Check that $e(P, \sigma) = e(P_K, H(m))$.

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Now looking for an accumulator... But we failed!

Thank you for your attention.