Solving sparse polynomial systems using Gröbner basis

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Resumé of the talk

Objective

Compute Gröbner basis faster by exploiting the sparsity of the supports of the polynomials.

We focus in the mixed case

The polynomials have different supports.

In this talk

- Algorithm to compute Gröbner basis over semigroup algebras.
- Under regularity assumptions, no reductions to zero.
- Algorithm and complexity bounds to solve 0-dim. square systems.
- Improvements for special cases (mixed multihomogeneous & unmixed).

Gröbner basics

- $\mathbb{K}[\boldsymbol{x}] = \mathbb{K}[x_1, \ldots, x_n]$, polynomial ring in n indeterminates over $\mathbb{K} \subset \mathbb{C}$.
- Polynomial $\rightarrow \sum_i c_i \boldsymbol{x}^\alpha \in \mathbb{K}[\boldsymbol{x}]$.
- Monomial $\rightarrow \boldsymbol{x}^\alpha$, for $\alpha \in \mathbb{N}^n$.

Monomial ordering $<$

Total order for monomials in $\mathbb{K}[\boldsymbol{x}]$ such that,

- The monomial 1 is the smallest: $\forall \boldsymbol{x}^\alpha \neq 1$, $1 < \boldsymbol{x}^\alpha$,
- Compatible with multiplication: for all $\boldsymbol{x}^\alpha, \boldsymbol{x}^\beta, \boldsymbol{x}^\gamma$,
$$\boldsymbol{x}^\alpha < \boldsymbol{x}^\beta \implies \boldsymbol{x}^\alpha \boldsymbol{x}^\gamma < \boldsymbol{x}^\beta \boldsymbol{x}^\gamma$$

- Lexicographical (lex) $y < x$,
$$1 < y < y^2 < \cdots < x < x\,y < x\,y^2 < \cdots < x^2 < x^2\,y < \ldots.$$
- Degree lexicographical $z < y < x$,
$$1 < z < y < x < z^2 < y\,z < y^2 < x\,z < x\,y < x^2 < \ldots.$$
- Degree reverse lexicographical order (grevlex) $z < y < x$,
$$1 < z < y < x < z^2 < y\,z < x\,z < y^2 < x\,y < x^2 < \ldots$$

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Leading monomial $\rightarrow$ Biggest monomial (wrt $>$) with non-zero coefficient.

Gröbner basis

A subset $G \subset I$ is a Gröbner basis of the ideal I wrt $>$, if and only if, for every $f \in I$, there is $g \in G$ such that $LM_>(g)$ divides $LM_>(f)$.

Computing Gröbner basis : Lazard's approach

Compute Gröbner basis for (f_1, f_2, f_3) in $\mathbb{K}[x, y]$ wrt Grevlex$(x > y)$,

$$\begin{cases} f_1 := & x + y + 1 \\ f_2 := & -x + y + 1 \\ f_3 := & x^2 + x\,y - y^2 + x + y + 1 \end{cases}$$

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We homogenize the system over $\mathbb{K}[x, y, z]$.

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Degree $d = 1$,

$$\left[\begin{array}{c|ccc} & x & y & z \\ F_1 & 1 & 1 & 1 \\ F_2 & -1 & 1 & 1 \end{array}\right]$$

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Degree $d = 2$,

$$\begin{array}{c} \\ z\,F_1 \\ y\,F_1 \\ x\,F_1 \\ z\,F_2 \\ y\,F_2 \\ x\,F_2 \\ F_3 \end{array} \left[\begin{array}{cccccc} x^2 & x\,y & y^2 & x\,z & y\,z & z^2 \\ & & & 1 & 1 & 1 \\ & 1 & 1 & & 1 & \\ 1 & 1 & & 1 & & \\ & & & -1 & 1 & 1 \\ & -1 & 1 & & 1 & \\ -1 & 1 & & 1 & & \\ 1 & 1 & -1 & 1 & 1 & 1 \end{array}\right]$$

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$$\begin{array}{r} \\ z\,F_1 \\ y\,F_1 \\ x\,F_1 \\ z\,F_2 + z\,F_1 \\ y\,F_2 + y\,F_1 \\ (x+y+z)\,F_2 - (x-y+z)\,F_1 \\ F_3 - (x - \frac{y}{2} + 1)F_1 + (\frac{y}{2} - 1)F_2 \end{array} \left[\begin{array}{cccccc} x^2 & x\,y & y^2 & x\,z & y\,z & z^2 \\ & & & 1 & 1 & 1 \\ & 1 & 1 & & 1 & \\ 1 & 1 & & 1 & & \\ & & & & 2 & 2 \\ & & 2 & & 2 & \\ & & & & & \\ & & & & & -1 \end{array}\right]$$

Computing Gröbner basis : Lazard's approach for Grevlex

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Gröbner basis of $\langle F_1, F_2, F_3 \rangle \rightarrow \{x + y + z, y + z, z^2\}$.

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Gröbner basis of $\langle F_1, F_2, F_3 \rangle \rightarrow \{x + y + z, y + z, z^2\}$.
Its dehomogenization $(z = 1)$ is a Gröbner basis of $\langle f_1, f_2, f_3 \rangle \rightarrow \{1\}$.

Complexity of Lazard's algorithm

Complexity depends on maximal degree. In generic coordinates,

$\rightarrow$ Castelnuovo-Mumford (CM) regularity of I.

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Regular sequence

$(F_1, \ldots, F_m)$ is a regular seq. $\Leftrightarrow \forall k \leq m$, F_k is regular in $\mathbb{K}[\boldsymbol{x}]/\langle F_1, \ldots, F_{k-1}\rangle$.

Macaulay bound

If $F_1, \ldots, F_m$ regular sequence $\rightarrow$ CM regularity $= \sum_{i=1}^{m} deg(f_i) - m + 1$

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Drawback: Many rows reduce to zero

	x^2	xy	y^2	xz	yz	z^2
$z\,F_1$				1	1	1
$y\,F_1$		1	1		1	
$x\,F_1$	1	1		1		
$z\,F_2$				-1	1	1
$y\,F_2 + y\,F_1$			2		2	
$(x+y+z)\,F_2 - (x-y+z)\,F_1$	0	0	0	0	0	0
$F_3 - (x - \frac{y}{2} + 1)F_1 + (\frac{y}{2} - 1)F_2$						-1

F5 criterion to detect trivial syzygies

$$\underbrace{(x - y + z)}_{F_2} F_1 - \underbrace{(x + y + z)}_{F_1} F_2 = 0 \longleftrightarrow \begin{array}{c}\text{Trivial syzygy} \\ (F_2, -F_1, 0)\end{array}$$

Trivial syzygy

Syzygy of $(F_1, \ldots, F_m) \rightarrow$

$$(H_1, \ldots, H_m) \in \mathbb{K}[\boldsymbol{x}]^m \text{ such that } \sum_i H_i F_i = 0.$$

Trivial $\rightarrow H_m = \cdots = H_{k+1} = 0,\ H_k \neq 0$, and

$$H_k \in \langle F_1, \ldots, F_{k-1} \rangle.$$

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Trivial syzygy

Syzygy of $(F_1, \ldots, F_m) \to (H_1, \ldots, H_m) \in \mathbb{K}[\boldsymbol{x}]^m$ such that $\sum_i H_i F_i = 0$.
Trivial $\to H_m = \cdots = H_{k+1} = 0,\ H_k \neq 0$, and $H_k \in \langle F_1, \ldots, F_{k-1} \rangle$.

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F5 criterion

If $x^\alpha \in LM_>(\langle F_1, \ldots, F_{k-1} \rangle)$, then the row $x^\alpha\, F_k$ reduces to zero.

$$\begin{array}{c} z\,F_1 \\ y\,F_1 \\ x\,F_1 \\ z\,F_2 \\ y\,F_2 \\ \hline x\,F_2 \\ \hline F_3 \end{array}
\left[\begin{array}{cccccc}
 & & & 1 & 1 & 1 \\
 & 1 & 1 & & 1 & \\
1 & 1 & & 1 & & \\
 & & & -1 & 1 & 1 \\
 & -1 & 1 & & 1 & \\ \hline
-1 & 1 & & 1 & & \\ \hline
1 & 1 & -1 & 1 & 1 & 1
\end{array}\right]
\to
\left[\begin{array}{cccccc}
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F5 criterion : Optimality

If $(F_1, \ldots, F_m)$ is a regular sequence $\implies$
F5 criterion detects all the reductions to zero.

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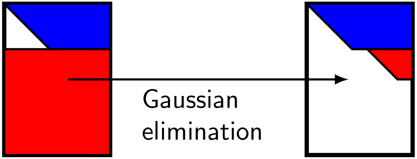
$$(J_k)_d := \text{Gaussian Elim.}\left((J_{k-1})_d \cup \{x^\alpha F_k : deg(x^\alpha) = d - deg(F_k)\} \right)$$

$(J_k)_d$



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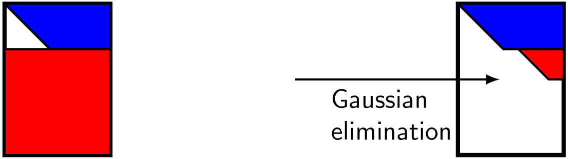

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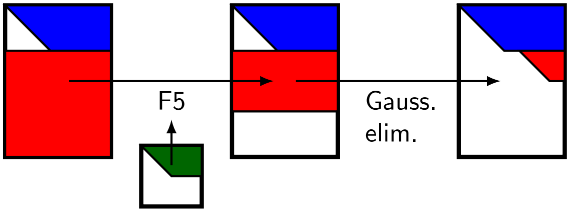

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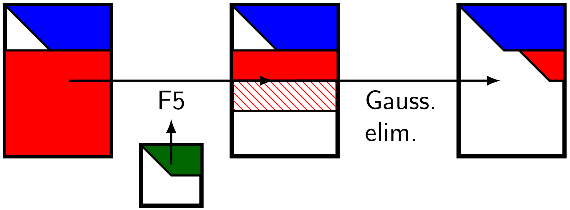

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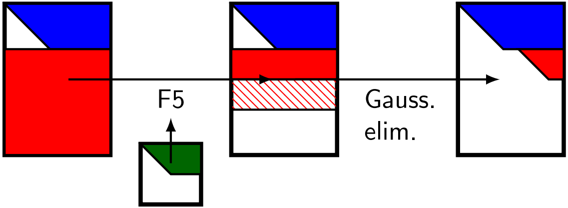

If $(F_1, \ldots, F_m)$ is a regular sequence $\Longrightarrow$ we skip every reduction to zero.

Summing up

Computing Gröbner basis

- Lazard's algorithm $\rightarrow$ Computes Gröbner basis using linear algebra.
- In generic coordinates, complexity $\rightarrow$ Castelnuovo-Mumford regularity.

- F5 criterion $\rightarrow$ Avoids trivial reductions to zero.
- If the system is a *regular sequence*.
  - F5 avoids every redundant computations.
  - Castelnuovo-Mumford regularity $\rightarrow$ Macaulay bound.

Summing up

Computing Gröbner basis for sparse systems

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  Generic coordinates destroy sparsity. Complexity unknown.

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Generic sparse systems are not *regular sequences*.

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Computing Gröbner basis for **sparse systems**

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- ✓ F5 criterion → Avoids trivial reductions to zero.
- ✗ ~~If the system is a *regular sequence*.~~

  Generic sparse systems are not *regular sequences*.
  - ✗ ~~F5 avoids every redundant computations.~~

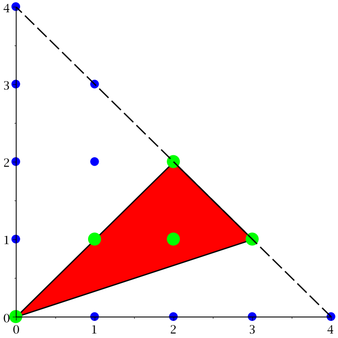
    F5 always misses reductions to zero.
  - ✗ ~~Castelnuovo-Mumford regularity → Macaulay bound.~~

    Castelnuovo-Mumford regularity unknown.

Sparse systems

- **Newton polytope** of $f = \sum_\alpha c_\alpha \boldsymbol{X}^\alpha \longrightarrow$ Convex hull of $\{\alpha : c_\alpha \neq 0\}$.
- Sparse system $\longrightarrow$ Newton polytopes of the polynomials are "small".

$$1 + xy + x^2y + x^2y^2 + x^3y = \mathbf{1} + \mathbf{0} \cdot x + \mathbf{0} \cdot y + \mathbf{0} \cdot x^2 + \boldsymbol{xy} + \mathbf{0} \cdot y^2 + \mathbf{0} \cdot x^3 + \boldsymbol{x^2y} + \mathbf{0} \cdot xy^2 + \mathbf{0} \cdot y^3 + \mathbf{0} \cdot x^4 + \boldsymbol{x^3y} + \boldsymbol{x^2y^2} + \mathbf{0} \cdot xy^3 + \mathbf{0} \cdot y^4$$

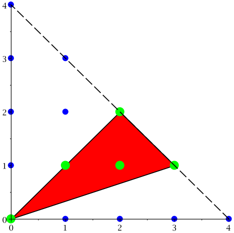


Sparse systems

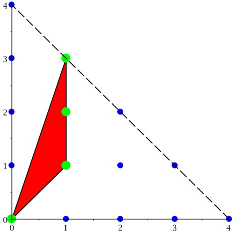
- **Newton polytope** of $f = \sum_\alpha c_\alpha \boldsymbol{X}^\alpha \longrightarrow$ Convex hull of $\{\alpha : c_\alpha \neq 0\}$.
- Sparse system $\longrightarrow$ Newton polytopes of the polynomials are "small".

- Unmixed sparse system $\longrightarrow$ Polynomials with equal Newton polytope.
  Mixed sparse system $\longrightarrow$ Different Newton polytope.

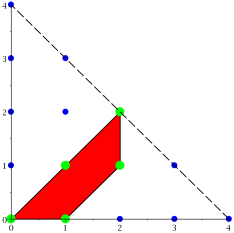
$1+xy+x^2y+x^2y^2+x^3y$



$1+xy+xy^2+xy^3$



$1+x+xy+x^2y+x^2y^2$



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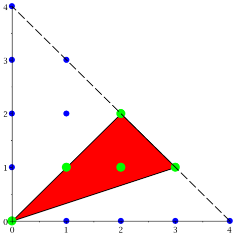
  This talk!

$1+xy+x^2y+x^2y^2+x^3y$



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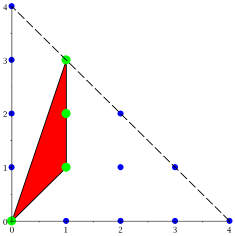

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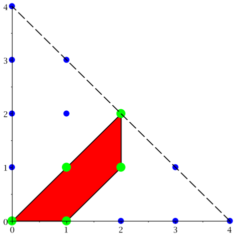
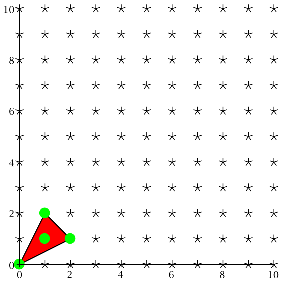


Previous work (Non-exhaustive!)

- Toric varieties
  [Demazure, 1970], [Hochster, 1971], [Satake, 1973], [Kempf, Knudsen, Mumford & Saint-Donat, 1973], [Miyake & Oda, 1975], [Ehlers, 1975], [Bernstein, 1975], [Kusnirenko, 1976] [Khovanskii, 1977], ...
  ... [Oda, 1988] ... [Fulton, 1993] ... [Cox, Little & Schenck, 2011]

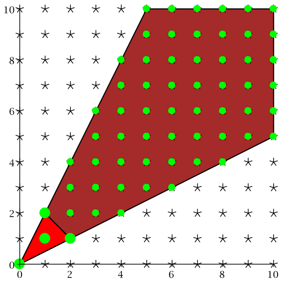
- Sparse resultant
  [Gelfand, Kapranov & Zelevinsky, 1990], [Kapranov, Sturmfels & Zelevinsky, 1992], [Sturmfels, 1993], [Pedersen & Sturmfels, 1993], [Gelfand, Kapranov & Zelevinsky, 1994], [Canny & Emiris, 1995], [D´Andrea, 2002], [D´Andrea & Sombra, 2013]

- Gröbner basis over semigroup algebras
  [Sturmfels, 1991], [Faugère, Spaenlehauer & Svartz, 2014],
  [B., Faugère & Tsigaridas, 2018]



$\mathbb{K}[S^h]$

Semigroup algebra $\mathbb{K}[S^h]$: Unmixed case



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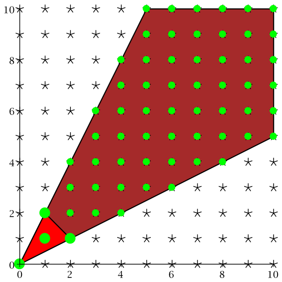
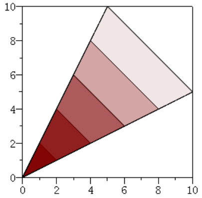


$\mathbb{K}[S^h]$

$\longrightarrow$



Graded algebra

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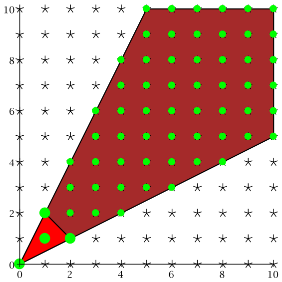

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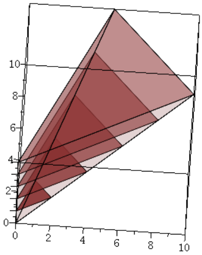

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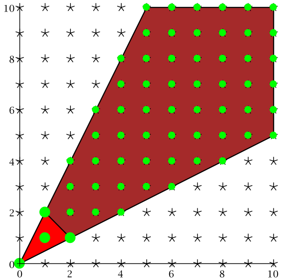

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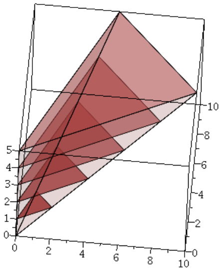
$\longrightarrow$



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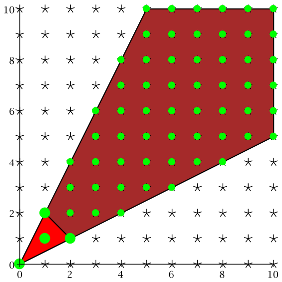

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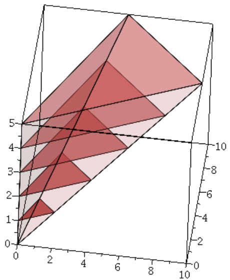
$\longrightarrow$



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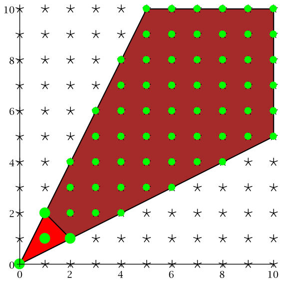

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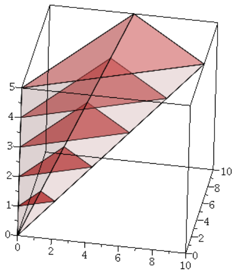

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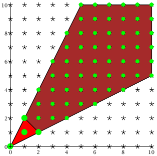
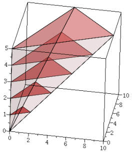


$\mathbb{K}[S^h]$

$\longrightarrow$



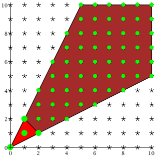
Graded algebra

[Faugère, Spaenlehauer & Svartz, 2014]

- GB over $\mathbb{K}[S^h] \rightarrow$ exists and finite.
- Algor. to compute GB over $\mathbb{K}[S^h]$ $\rightarrow$ Lazard's algorithm + F5.
- If the homogenization of $f_1, \ldots, f_m$ over $\mathbb{K}[S^h]$ is a *regular sequence*,
  - No redundant computations (F5 criterion).
  - ? 0-dim systems $\rightarrow$ Complexity bounds (CM regularity).

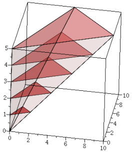
Generic **unmixed** systems $\rightarrow$ homogenization *regular*.

Semigroup algebra $\mathbb{K}[S^h]$: Unmixed case



$\mathbb{K}[S^h]$

$\longrightarrow$



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Generic **unmixed** systems $\rightarrow$ homogenization *regular*.

Generic **mixed** systems $\rightarrow$ homogenization **NOT** *regular*.

Relaxing the regular sequence condition

We need a new algorithm

Generic **mixed** systems are **NOT** *regular sequences*...

... but they behave as them for "big degrees".

Idea

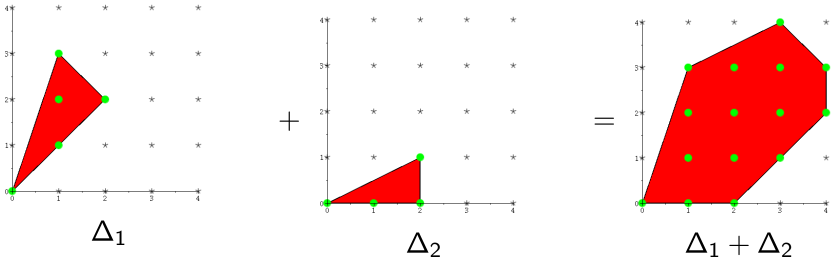
Check what happens at specific multidegrees.

- Consider multigrading for $\mathbb{K}[S^h]$ related to the different polytopes.
- Characterize the optimality of F5 at a multideg. $\rightarrow$ Koszul complex.
- Do not compute in every multidegree,
  only consider the ones where we can predict the syzygies.

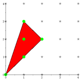
Warning

This approach **does not** make the systems regular sequences.

Minkowski sum

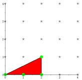


Minkowski sum



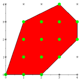
Δ_1

$+$



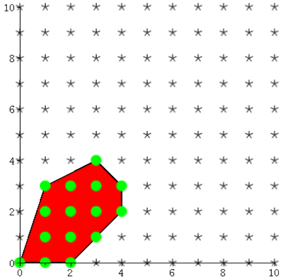
Δ_2

$=$



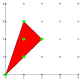
$\Delta_1 + \Delta_2$

Semigroup algebra



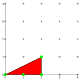
$\mathbb{K}[S^h]$

Minkowski sum



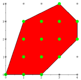
Δ_1

$+$



Δ_2

$=$



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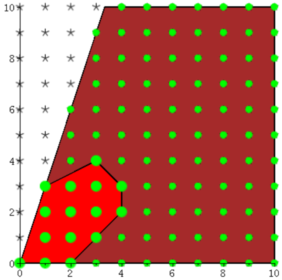
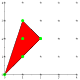
Semigroup algebra



$\mathbb{K}[S^h]$

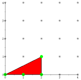
Semigroup algebra $\mathbb{K}[S^h]$: Mixed case

Minkowski sum



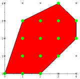
Δ_1

$+$



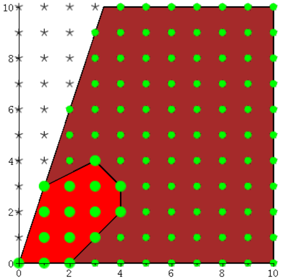
Δ_2

$=$



$\Delta_1 + \Delta_2$

Semigroup algebra



$\mathbb{K}[S^h]$

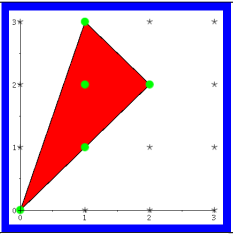
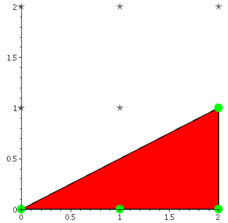
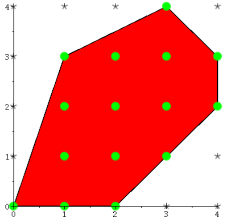
$\mathbb{K}[S^h]$ multigraded by $\mathbb{N}^m$

$$X^\alpha \in \mathbb{K}[S^h]_{(d_1,\ldots,d_m)}$$
$$\updownarrow$$
$$\alpha \in (d_1\Delta_1 + \cdots + d_m\Delta_m) \cap \mathbb{Z}^n$$

Semigroup algebra $\mathbb{K}[S^h]$: Mixed case

$\mathbb{K}[S^h]$ is multigraded by $\mathbb{N}^2$ wrt Δ_1, Δ_2,

$$X^\alpha \in \mathbb{K}[S^h]_{(d_1,d_2)} \longleftrightarrow \alpha \in (d_1\Delta_2 + d_2\Delta_2) \cap \mathbb{Z}^n$$

$+$		Δ_1
	(1,0)	
Δ_2		

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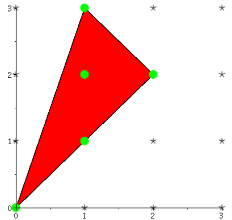
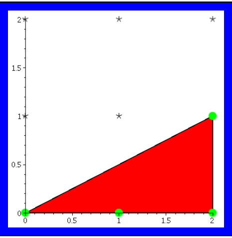
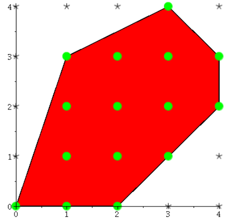
+		Δ_1
	(0,1)	
Δ_2		

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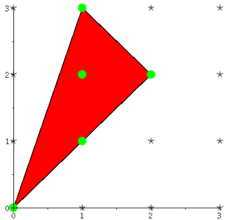
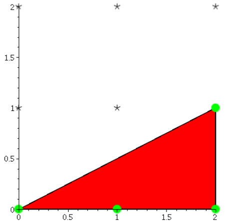
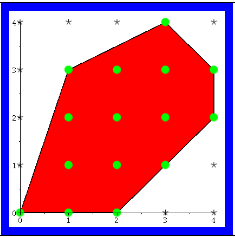
+		Δ_1
	(1,1)	
Δ_2		

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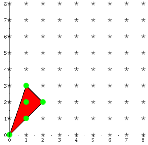
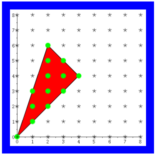
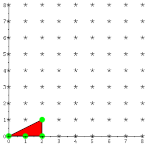
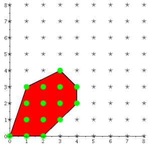
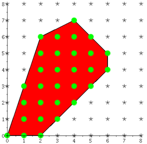
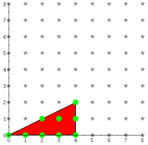
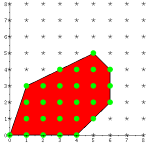
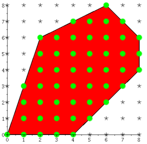
+		Δ_1	$2\Delta_1$
	(2,0)		
Δ_2			
$2\Delta_2$			

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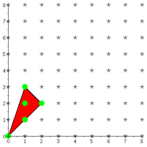
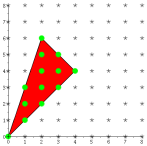
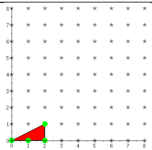
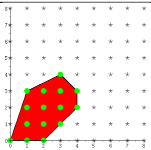
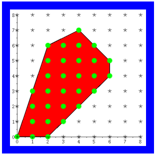
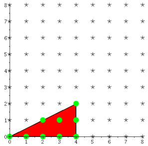
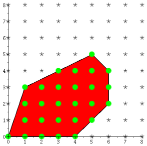
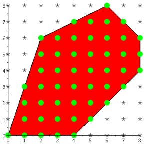
+		Δ_1	$2\Delta_1$
	(2,1)		
Δ_2			
$2\Delta_2$			

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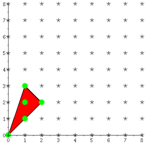
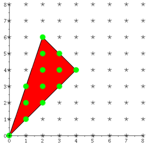
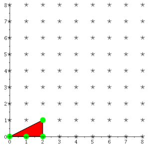
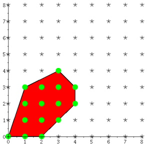
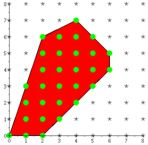
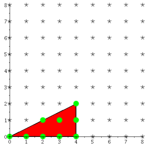
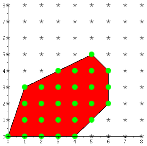
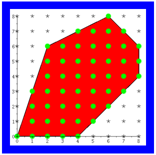
$+$		Δ_1	$2\Delta_1$
	(2,2)		
Δ_2			
$2\Delta_2$			

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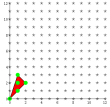
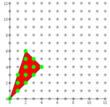
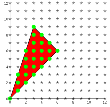
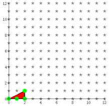
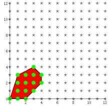
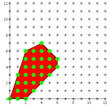
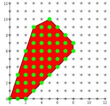
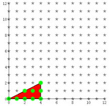
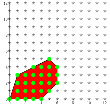
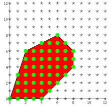
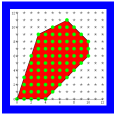
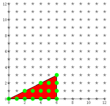
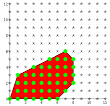
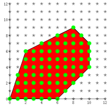
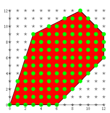
+		Δ_1	$2\Delta_1$	$3\Delta_1$
	(2,3)			
Δ_2				
$2\Delta_2$				
$3\Delta_2$				

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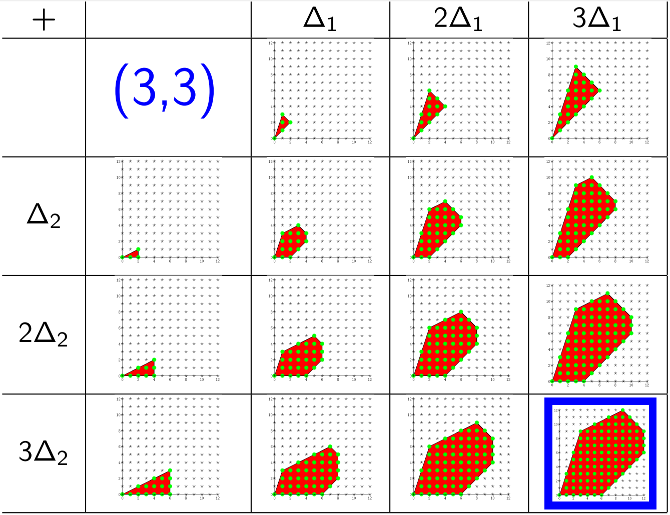

Koszul F5 criterion

Koszul complex,

$$\mathcal{K}(F_1, \ldots, F_k) : 0 \to (\mathcal{K}_k) \xrightarrow{\delta_k} \ldots \xrightarrow{\delta_2} (\mathcal{K}_1) \xrightarrow{\delta_1} (\mathcal{K}_0) \to 0,$$

- Matrices in Lazard's algo. represent
  $\delta_1(G_1, \ldots, G_k) = \sum_i G_i F_i$
- Trivial syzygies $\leftrightarrow \mathrm{Im}(\delta_2)$
  - F5 is correct $\Leftrightarrow \mathrm{Im}(\delta_2) \subset \mathtt{Ker}(\delta_1)$
  - F5 is optimal $\Leftrightarrow \mathrm{Im}(\delta_2) = \mathtt{Ker}(\delta_1)$

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Koszul F5 criterion

If, for each $k \leq m$, $\mathcal{K}(F_1, \ldots, F_k)$ is exact at multidegree d,
$\implies$ every syzygy of $(F_1, \ldots, F_m)$ of multidegree d is trivial.

$(F_1, \ldots, F_m)$ is (sparse) regular

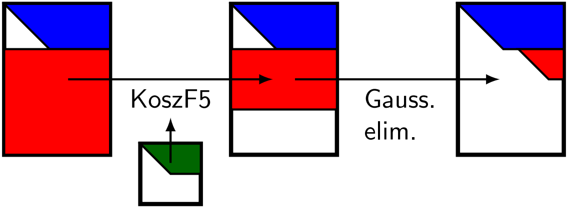
For each $k \leq m$, $\mathcal{K}(F_1, \ldots, F_k)$ is exact at multideg. d, for $d \geq \sum_{i \leq k} mdeg(F_i)$.

Computing GB for mixed systems

Algorithm: For big multidegree d, compute $\texttt{triangBasis}((\langle F_1, \ldots, F_m \rangle)_d)$.

$\texttt{triangBasis}((\langle F_1, \ldots, F_k \rangle)_d) \leftarrow$ Gaussian elimination of
$\texttt{triangBasis}((\langle F_1, \ldots, F_{k-1} \rangle)_d) \cup \{X^\alpha F_k : \texttt{KoszulF5}_k(X^\alpha)\}$.

$\texttt{KoszulF5}_k(X^\alpha) \leftarrow$ Skip X^α if it is a leading monomial of
$\texttt{triangBasis}((\langle F_1, \ldots, F_{k-1} \rangle)_{d-mdeg(F_k)})$.



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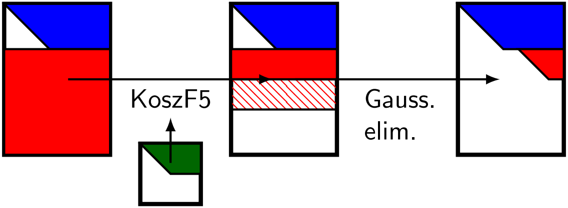

If multidegree $d \geq \sum_{i \leq k} mdeg(F_i)$, and $(F_1, \ldots, F_m)$ is (sparse) regular $\Longrightarrow$
No reductions to zero.

Solving sparse polynomials systems over $(\mathbb{C}^*)^n$

Solving sparse polynomials systems over $(\mathbb{C}^*)^n$

Consider square system $(f_1, \ldots, f_n)$ with polytopes $\Delta_1, \ldots, \Delta_n$.

BKK bound

Number of solutions of $(f_1, \ldots, f_n)$ over $(\mathbb{C}^*)^n \leq$ Mixed volume of $\Delta_1, \ldots, \Delta_n$.

- $K[S^h] \leftarrow$ Semigroup algebra of $\Delta_0, \Delta_1, \ldots, \Delta_n$. ($\Delta_0$ is n-standard simplex)
- $(F_1, \ldots, F_n) \leftarrow$ homogenization over $\mathbb{K}[S^h]$. $(multideg(F_i) = \boldsymbol{e}_i \in \mathbb{N}^{n+1})$

If BKK bound is tight and $(F_1, \ldots, F_n)$ is (sparse) regular,
We can compute GB of $\langle f_1, \ldots, f_n \rangle : \langle \prod_i x_i \rangle^\infty$ in

$$O\left(2^{n+1}\left(\#\left(\sum_{i=0}^{n} \Delta_i \cap \mathbb{Z}^n\right)\right)^{\omega} + n\, MV(\Delta_1, \ldots, \Delta_n)^3\right).$$

Similar complexity to resultant based methods, i.e. [Canny & Emiris, 1995],
$\rightarrow$ but clear assumptions and correct for non-radical ideals.

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Solving sparse polynomials systems over $(\mathbb{C}^*)^n$

For each linear $f_0 \in \mathbb{K}[\boldsymbol{x}]$, consider $M(f_0)$,

$$\begin{array}{r} \texttt{triangBasis}((\langle F_1, \dots, F_n \rangle)_{(1,1,\dots,1)}) \left\{ \begin{array}{c} \\ \\ \end{array} \right. \\ \{X^\alpha F_0 : X^\alpha \in L\} \left\{ \begin{array}{c} \\ \end{array} \right. \end{array} \left[\begin{array}{c||c} M_{1,1}(f_0) & \overbrace{M_{1,2}(f_0)}^{L} \\ \hline\hline M_{2,1}(f_0) & M_{2,2}(f_0) \end{array} \right]$$

$F_0 \in \mathbb{K}[S^h]_{(1,0,\dots,0)} \leftarrow$ Homogenization of f_0.

$\boldsymbol{L} \leftarrow$ Monomials not in $\texttt{LM}\big(\texttt{triangBasis}((\langle F_1, \dots, F_n \rangle)_{(0,1,\dots,1)})\big)$.

The matrix $M(f_0)$ is square, with (row/column) dimension $\#\left(\sum_{i \geq 0} \Delta_i \cap \mathbb{Z}^n\right)$.

Schur complement of $M(f_0) \leftrightarrow$ mult. map of f_0 in $\mathbb{K}[x_1^{\pm}, \dots, x_n^{\pm}]/\langle f_1, \dots, f_n \rangle$

$$M_{2,2}^c(f_0) := (M_{2,2} - M_{2,1}\, M_{1,1}^{-1}\, M_{1,2})(f_0).$$

Improving the bounds, Multigraded Castelnuovo-Mumford regularity

Improving the bounds for solving 0-dim systems

We rely on

- When the Koszul complex is exact $\rightarrow$ Avoid reduction to zero
- At the multigraded CM regularity $\rightarrow$ Recover multiplication maps

[Maclagan & Smith, 2004], [Botbol & Chardin, 2017]

Under reg. assumptions, bounds for exactness of Koszul complex $\implies$ bounds for multigraded Castelnuovo-Mumford regularity.

New complexity bounds

- Unmixed sparse systems.
- Mixed multihomogeneous systems.

The complexity of solving unmixed sparse systems

- Algorithm to solve, over $(\mathbb{C}^*)^n$, 0-dimensional square **unmixed sparse systems**, which performs **no reduction to zero**.
- **Complexity bounds** in terms of **Castelnuovo-Mumford regularity**.

[Bruns, Gubeladze & Trung, 1997]

Let t be the smallest integer such that $t \cdot \Delta$ has an integer interior point. Then, the Castelnuovo-Mumford regularity of $K[S^h]$ is $n - t + 1$.

Maximal degree

New bound	General bound (no assumptions)
$\sum_i mdeg(f_i) + 1 - (t - 1)$	$\sum_i mdeg(f_i) + 1$

The complexity of solving multihomogeneous systems

[B., Faugère & Tsigaridas, 2018]

- Algorithm to solve, over $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, 0-dimensional square **mixed multihomogeneous** systems, which performs **no reduction to zero**.
- **Complexity bounds** in terms of **Multihomogeneous Macaulay bound**.

[Botbol & Chardin, 2017]

We exploit their bounds for the multigraded Castenuovo-Mumford regularity.

Multihomogeneous Macaulay bound → Generalization of Macaulay bound

Macaulay bound	Multihomogeneous Macaulay bound
$\sum_{i=1}^{n} deg(f_i) - n + 1$	$\sum_{i=1}^{n_1+\cdots+n_s} multideg(f_i) - (n_1, \ldots, n_s) + \bar{1}$

The general bound is $\sum_i multideg(f_i) + \bar{1}$

Summing-up

Tools

- Exploit sparseness $\rightarrow$ Gröbner basis for semigroup algebras.
- No reductions to zero $\rightarrow$ Exactness of the Koszul complex.
- Complexity bounds $\rightarrow$ Multigraded Castelnuovo-Mumford regularity.

Results

- Algorithm to compute GB for semigroup algebras.
- Regularity assumptions for **mixed** systems $\rightarrow$ no reductions to zero.
- F5 criterion related to Koszul complex, not to regular sequences.
- Algorithm and complexity bounds to solve 0-dim. square systems.
- Improvements for special cases (mixed multihomogeneous & unmixed).

Perspectives

- Exploit the combinatorial structures of the polytopes.

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Thank you!