Computing Gröbner Bases – a short overview

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Conventions

- $\mathscr{R} = \mathscr{K}[x_1, \ldots, x_n]$, $\mathscr{K}$ field, $<$ well-ordering on $\mathrm{Mon}(x_1, \ldots, x_n)$
- $f \in \mathscr{R}$ can be represented in a unique way by $<$.
  $\Rightarrow$ Definitions as $\mathrm{lc}(f)$, $\mathrm{lm}(f)$, and $\mathrm{lt}(f)$ make sense.
- An ideal I in $\mathscr{R}$ is an additive subgroup of $\mathscr{R}$ such that for $f \in I$, $g \in \mathscr{R}$ it holds that $fg \in I$.
- $G = \{g_1, \ldots, g_s\} \subset \mathscr{R}$ is a Gröbner basis for $I = \langle f_1, \ldots, f_m \rangle$ w.r.t. $<$

$$:\Longleftrightarrow$$

$$G \subset I \text{ and } L_<(G) = L_<(I)$$

Buchberger's criterion

S-polynomials

Let $f \neq 0, g \neq 0 \in \mathscr{R}$ and let $\lambda = \operatorname{lcm}\left(\operatorname{lt}(f), \operatorname{lt}(g)\right)$ be the least common multiple of $\operatorname{lt}(f)$ and $\operatorname{lt}(g)$. The **S-polynomial** between f and g is given by

$$\operatorname{spol}(f, g) := \frac{\lambda}{\operatorname{lt}(f)} f - \frac{\lambda}{\operatorname{lt}(g)} g.$$

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Buchberger's criterion [5]

Let $I = \langle f_1, \ldots, f_m \rangle$ be an ideal in $\mathscr{R}$. A finite subset $G \subset \mathscr{R}$ is a **Gröbner basis for** I if $G \subset I$ and for all $f, g \in G: \operatorname{spol}(f,g) \xrightarrow{G} 0$.

Buchberger's algorithm

Input: Ideal $I = \langle f_1, \ldots, f_m \rangle$
Output: Gröbner basis G for I

1. $G \leftarrow \emptyset$
2. $G \leftarrow G \cup \{f_i\}$ for all $i \in \{1, \ldots, m\}$
3. Set $P \leftarrow \{\text{spol}(f_i, f_j) \mid f_i, f_j \in G, i > j\}$

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4. Choose $p \in P$, $P \leftarrow P \setminus \{p\}$
   (a) If $p \xrightarrow{G} 0$ ▶▶ **no new information**
   Go on with the next element in P.
   (b) If $p \xrightarrow{G} q \neq 0$ ▶▶ **new information**
   Build new S-pair with q and add them to P.
   Add q to G.
   Go on with the next element in P.
5. When $P = \emptyset$ we are done and G is a Gröbner basis for I.

How to improve computations?

- Modular computations (modStd et al.)
- Predict zero reductions (Buchberger, Gebauer-Möller, Möller-Mora-Traverso, Faugère.)
- Sort pair set (Buchberger, Giovini et al., Möller et al.)
- Homogenize: d-Gröbner bases
- Change of ordering (FGLM, Gröbner Walk)
- Linear Algebra: Gaussian Elimination (Lazard, Faugère)
- Sparse Gröbner Bases: Use sparsity and exploit Newton polygons (Faugère, Spaenlehauer, Svartz)
- ...

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- Linear Algebra: Gaussian Elimination (Lazard, Faugère)

- Predicting zero reductions

- Fast linear algebra for computing Gröbner bases

How to detect zero reductions in advance?

Let $I = \langle g_1, g_2 \rangle \in \mathbb{Q}[x, y, z]$ and let $<$ denote the reverse lexicographical ordering. Let

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$$\begin{aligned}\mathrm{spol}(g_2, g_1) &= xg_2 - yg_1 = \mathbf{xy^2} - xz^2 - \mathbf{xy^2} + yz^2 \\ &= -xz^2 + yz^2.\end{aligned}$$

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$$\mathrm{spol}(g_3, g_1) = \mathbf{xyz^2} - y^2z^2 - \mathbf{xyz^2} + z^4 = -y^2z^2 + z^4.$$

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We can reduce further using $z^2 g_2$:

$$\boxed{-y^2z^2 + z^4 + y^2z^2 - z^4 = 0.}$$

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For all $u \in \operatorname{support}(\operatorname{lot}(g_3))$ we can reduce with ug_2:

$$\begin{aligned}&\Longrightarrow \operatorname{lt}(g_2)\operatorname{lot}(g_3) - \mathbf{g_2}\operatorname{lot}(\mathbf{g_3}) - \operatorname{lt}(g_3)\operatorname{lot}(g_2)\\ &= -\operatorname{lot}(g_2)\operatorname{lot}(g_3) - \operatorname{lt}(g_3)\operatorname{lot}(g_2)\\ &= -g_3\operatorname{lot}(g_2).\end{aligned}$$

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&= -\,\mathrm{lot}\left(g_2\right)\mathrm{lot}\left(g_3\right) - \mathrm{lt}\left(g_3\right)\mathrm{lot}\left(g_2\right)\\
&= -\,g_3\mathrm{lot}\left(g_2\right).
\end{aligned}$$

So we can reduce this to zero by vg_3 for all $v \in \mathrm{support}\left(\mathrm{lot}\left(g_2\right)\right)$.

Buchberger's criteria

Product criterion [6, 7]

If $\operatorname{lcm}(\operatorname{lt}(f), \operatorname{lt}(g)) = \operatorname{lt}(f)\operatorname{lt}(g)$ then $\operatorname{spol}(f, g) \xrightarrow{\{f,g\}} 0$.

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$\Longrightarrow$ We can rewrite $\operatorname{spol}(g_3, g_2)$:

$$\operatorname{spol}(g_3, g_2) = y\,\underbrace{\operatorname{spol}(g_3, g_1)}_{\xrightarrow{G} 0} - z^2\,\underbrace{\operatorname{spol}(g_2, g_1)}_{\xrightarrow{G} -g_3} = y\,(yg_3 - z^2 g_1) - z^2\,(xg_2 - yg_1)$$

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Standard representations of $\operatorname{spol}(g_2, g_1)$ and $\operatorname{spol}(g_3, g_1)$
$\Longrightarrow$ Standard representation of $\operatorname{spol}(g_3, g_2)$.

Chain criterion [8]

Let $f, g, h \in \mathscr{R}$, $G \subset \mathscr{R}$ finite. If

1. $\operatorname{lt}(h) \mid \operatorname{lcm}(\operatorname{lt}(f), \operatorname{lt}(g))$, and
2. $\operatorname{spol}(f, h)$ and $\operatorname{spol}(h, g)$ have a standard representation w.r.t. G respectively,

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Note

Do not remove too much information! If $\lambda = 1$ and

$$\operatorname{spol}(f, g) = \lambda \operatorname{spol}(f, h) + \sigma \operatorname{spol}(h, g),$$

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How to combine Product and Chain criterion?

Gebauer-Möller installation [32]

We add a new element h to G and generate new pairs $P' := \{(f, h) \mid f \in G\}$.

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2. Fix $(f,h) \in P'$. If $(g,h) \in P' \setminus \{(f,h)\}$ s.t.
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4. If $(f,h) \in P'$ s.t. $\mathrm{lcm}(\mathrm{lt}(f), \mathrm{lt}(h)) = \mathrm{lt}(f)\,\mathrm{lt}(h)$
   $\Longrightarrow$ Remove (f,h) from P'. [Product criterion done]

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How to get rid of this useless computation?

Use more structure of $I \Longrightarrow$ **Signatures**

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4. A **signature** of f is given by $\mathfrak{s}(f) = \mathrm{lt}_{\prec}(\alpha)$ where $f = \overline{\alpha}$.
5. An element $\alpha \in \mathscr{R}^m$ such that $\overline{\alpha} = 0$ is called a **syzygy**.

Our example again – with signatures and $\prec_{pot}$

$$g_1 = xy - z^2,\ \mathfrak{s}(g_1) = e_1,$$
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$$\Rightarrow \mathfrak{s}\,(\mathrm{spol}(g_3, g_1)) = y\,\mathfrak{s}(g_3) = xye_2.$$

Note that $\mathfrak{s}\,(\mathrm{spol}(g_3, g_1)) = xy\,e_2$ and $\mathrm{lm}(g_1) = xy$.

Think in the module

$\alpha \in \mathscr{R}^m \implies$ polynomial $\overline{\alpha}$ with $\operatorname{lt}(\overline{\alpha})$, signature $\mathfrak{s}(\alpha) = \operatorname{lt}(\alpha)$

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Remark

In the following we need one detail from signature-based Gröbner Basis computations:

We pick from P by increasing signature.

Signature-based criteria

$$\mathfrak{s}(\alpha) = \mathfrak{s}(\beta) \implies \text{Compute 1, remove 1.}$$

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Sketch of proof

1. $\mathfrak{s}(\alpha - \beta) \prec \mathfrak{s}(\alpha), \mathfrak{s}(\beta)$.
2. All S-pairs are handled by increasing signature.
   $\Rightarrow$ All relatons $\prec \mathfrak{s}(\alpha)$ are known:

$$\alpha = \beta + \text{ elements of smaller signature}$$

$\square$

Signature-based criteria

S-pairs in signature T

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What are all possible configurations to reach signature T?

Signature-based criteria

S-pairs in signature T

$$\mathfrak{R}_T = \Big\{ a\alpha \mid \alpha \text{ handled by the algorithm and } \mathfrak{s}(a\alpha) = T \Big\}$$

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S-pairs in signature T

$$\mathfrak{R}_T = \left\{ a\alpha \mid \alpha \text{ handled by the algorithm and } \mathfrak{s}(a\alpha) = T \right\}$$

What are all possible configurations to reach signature T?

Define an order on $\mathfrak{R}_T$ and choose the maximal element.

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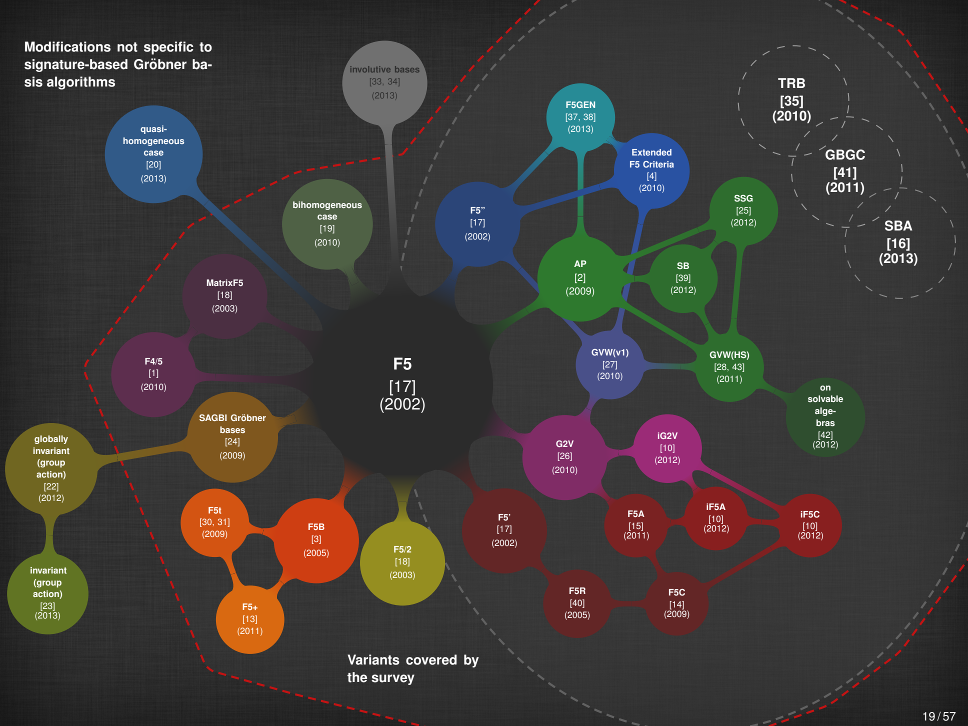
Revisiting our example with $\prec_{\text{pot}}$

$$\mathfrak{s}\left(\text{spol}(g_3, g_1)\right) = xye_2$$

$$\left.\begin{array}{l} g_1 = xy - z^2 \\ g_2 = y^2 - z^2 \end{array}\right\} \Rightarrow \text{psyz}(g_2, g_1) = g_1 e_2 - g_2 e_1 = xye_2 + \dots$$



Modifications not specific to signature-based Gröbner basis algorithms
involutive bases [33, 34] (2013)
quasi-homogeneous case [20] (2013)
bihomogeneous case [19] (2010)
MatrixF5 [18] (2003)
F4/5 [1] (2010)
SAGBI Gröbner bases [24] (2009)
globally invariant (group action) [22] (2012)
invariant (group action) [23] (2013)
F5t [30, 31] (2009)
F5B [3] (2005)
F5+ [13] (2011)
F5/2 [18] (2003)
F5 [17] (2002)
F5GEN [37, 38] (2013)
Extended F5 Criteria [4] (2010)
F5" [17] (2002)
AP [2] (2009)
SSG [25] (2012)
SB [39] (2012)
GVW(v1) [27] (2010)
GVW(HS) [28, 43] (2011)
on solvable algebras [42] (2012)
G2V [26] (2010)
iG2V [10] (2012)
F5' [17] (2002)
F5A [15] (2011)
iF5A [10] (2012)
iF5C [10] (2012)
F5R [40] (2005)
F5C [14] (2009)
TRB [35] (2010)
GBGC [41] (2011)
SBA [16] (2013)
Variants covered by the survey

zero reductions (Singular-4-0-0, $\mathbb{F}_{32003}$)

Benchmark	STD	SBA $\prec_{\text{pot}}$	SBA $\prec_{\text{d-pot}}$	SBA $\prec_{\text{lt}}$
cyclic-8	4,284	243	243	671
cyclic-8-h	5,843	243	243	671
eco-11	3,476	0	749	749
eco-11-h	5,429	502	502	749
katsura-11	3,933	0	0	348
katsura-11-h	3,933	0	0	348
noon-9	25,508	0	0	682
noon-9-h	25,508	0	0	682
Random$(11, 2, 2)$	6,292	0	0	590
HRandom$(11, 2, 2)$	6,292	0	0	590
Random$(12, 2, 2)$	13,576	0	0	1,083
HRandom$(12, 2, 2)$	13,576	0	0	1,083

Time in seconds (Singular-4-0-0, $\mathbb{F}_{32003}$)

Benchmark	STD	SBA $\prec_{\text{pot}}$	SBA $\prec_{\text{d-pot}}$	SBA $\prec_{\text{lt}}$
cyclic-8	32.480	44.310	100.780	38.120
cyclic-8-h	38.300	35.770	98.440	32.640
eco-11	28.450	3.450	27.360	13.270
eco-11-h	20.630	11.600	14.840	7.960
katsura-11	54.780	35.720	31.010	11.790
katsura-11-h	51.260	34.080	32.590	17.230
noon-9	29.730	12.940	14.620	15.220
noon-9-h	34.410	17.850	20.090	20.510
$\text{Random}(11, 2, 2)$	267.810	77.430	130.400	28.640
$\text{HRandom}(11, 2, 2)$	22.970	14.060	39.320	3.540
$\text{Random}(12, 2, 2)$	2,069.890	537.340	1,062.390	176.920
$\text{HRandom}(12, 2, 2)$	172.910	112.420	331.680	22.060

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▼
Generate all possible principal syzygies with α.
(e.g. **GVW**)

S-pair fulfilling Product criterion not detected by Rewrite criterion
▼
Add one corresponding syzygy.
(e.g. **SBA** in **Singular**)

Experimental results

Implementation done in **Singular** [9]

Benchmark	STD	SBA $\prec_{pot}$	SBA $\prec_{lt}$	
	ZR	ZR	ZR	ZR / PC
cyclic-8	4284	243	671	671 / 0
cyclic-8-h	5843	243	671	671 / 0
eco-11	3476	0	749	749 / 0
eco-11-h	5429	502	749	718 / 0
katsura-11	3933	0	348	304 / 0
katsura-11-h	3933	0	348	304 / 0
noon-9	25508	0	682	646 / 0
noon-9-h	25508	0	682	646 / 0
binomial-6-2	21	6	15	8 / 7
binomial-6-3	20	13	15	9 / 6
binomial-7-3	27	24	21	21 / 0
binomial-7-4	41	16	19	16 / 3
binomial-8-3	53	23	27	27 / 0
binomial-8-4	40	31	26	26 / 0

And what's about SBA using $\prec_{\text{pot}}$?

Conjecture [11]
Every S-polynomial fulfilling the Product criterion is also detected by the Rewrite criterion in **SBA** using $\prec_{\text{pot}}$.

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- We checked several million examples, all fulfilling the conjecture.
- Until now we cannot prove this.

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Every S-polynomial fulfilling the Product criterion is also detected by the Rewrite criterion in **SBA** using $\prec_{\text{pot}}$.

- We checked several million examples, all fulfilling the conjecture.
- Until now we cannot prove this.

Ongoing work:

1. Describe in detail the connection between our conjecture and Moreno-Socías conjecture [36].
2. Try to exploit even more algebraic structures for predicting zero reductions.

- Predicting zero reductions

- Fast linear algebra for computing Gröbner bases

Buchberger's algorithm - revisited

Input: Ideal $I = \langle f_1, \ldots, f_m \rangle$
Output: Gröbner basis G for I

1. $G \leftarrow \emptyset$
2. $G \leftarrow G \cup \{f_i\}$ for all $i \in \{1, \ldots, m\}$
3. Set $P \leftarrow \{\mathrm{spol}(f_i, f_j) \mid f_i, f_j \in G, i > j\}$
4. Choose $p \in P$, $P \leftarrow P \setminus \{p\}$
   (a) If $p \xrightarrow{G} 0$ ►► **no new information**
   Go on with the next element in P.
   (b) If $p \xrightarrow{G} q \neq 0$ ►► **new information**
   Build new S-pair with q and add them to P.
   Add q to G.
   Go on with the next element in P.
5. When $P = \emptyset$ we are done and G is a Gröbner basis for I.

Faugère's F4 algorithm

Input: Ideal $I = \langle f_1, \ldots, f_m \rangle$
Output: Gröbner basis G for I

1. $G \leftarrow \emptyset$
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4. $d \leftarrow 0$
5. while $P \neq \emptyset$:

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3. Set $P \leftarrow \{(af, bg) \mid f, g \in G\}$
4. $d \leftarrow 0$
5. while $P \neq \emptyset$:
   (a) $d \leftarrow d + 1$
   (b) $P_d \leftarrow$ **Select** (P), $P \leftarrow P \setminus P_d$
   (c) $L_d \leftarrow \{af, bg \mid (af, bg) \in P_d\}$
   (d) $L_d \leftarrow$ **Symbolic Preprocessing**(L_d, G)
   (e) $F_d \leftarrow$ **Reduction**(L_d, G)
   (f) for $h \in F_d$:
   - If $\operatorname{lt}(h) \notin L(G)$ (all other h are "useless"):
     - $P \leftarrow P \cup \{\text{new pairs with } h\}$
     - $G \leftarrow G \cup \{h\}$
6. Return G

Differences to Buchberger

1. Select a subset P_d of P, not only one element.
2. Do a symbolic preprocessing:
   Search and store reducers, but do not reduce.
3. Do a full reduction of P_d at once:
   Reduce a subset of $\mathscr{R}$ by a subset of $\mathscr{R}$

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If **Select** (P) selects only 1 pair F4 is just Buchberger's algorithm.
Usually one chooses the normal selection strategy,
i.e. all pairs of lowest degree.

Symbolic preprocessing

Input: L, G finite subsets of $\mathscr{R}$
Output: a finite subset of $\mathscr{R}$

1. $F \leftarrow L$
2. $D \leftarrow L(F)$ (S-pairs already reduce lead terms)
3. while $T(F) \neq D$:
   (a) Choose $m \in T(F) \setminus D$, $D \leftarrow D \cup \{m\}$.
   (b) If $m \in L(G) \Rightarrow \exists\, g \in G$ and $\lambda \in \mathscr{R}$ such that $\lambda\,\mathrm{lt}(g) = m$
   $\triangleright\; F \leftarrow F \cup \{\lambda g\}$
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We optimize this soon!

Reduction

Input: L, G finite subsets of $\mathscr{R}$
Output: a finite subset of $\mathscr{R}$

1. $M \leftarrow$ Macaulay matrix of L
2. $M \leftarrow$ Gaussian Elimination of M (Linear algebra)
3. $F \leftarrow$ polynomials from rows of M
4. Return F

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Macaulay matrix

columns	$\hat{=}$	monomials (sorted by monomial order $<$)
rows	$\hat{=}$	coeffs of polynomials in L

Example: Cyclic-4

$\mathscr{R} = \mathbb{Q}[a, b, c, d]$, $<$ denotes DRL and we use the normal selection strategy for **Select** (P). $I = \langle f_1, \ldots, f_4 \rangle$, where

$$\begin{aligned} f_1 &= abcd - 1, \\ f_2 &= abc + abd + acd + bcd, \\ f_3 &= ab + bc + ad + cd, \\ f_4 &= a + b + c + d. \end{aligned}$$

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Let us do symbolic preprocessing:

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T(L_1) &= \{ab, b^2, bc, ad, bd, cd, d^2\} \\
L_1 &= \{f_3, bf_4, df_4\}
\end{aligned}
$$

$b^2 \notin L(G)$, $bc \notin L(G)$, $d\,\mathrm{lt}(f_4) = ad$,

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$b^2 \notin L(G)$, $bc \notin L(G)$, $d\,\mathrm{lt}(f_4) = ad$, all others also $\notin L(G)$,

Example: Cyclic-4

Now reduction:
Convert polynomial data L_1 to Macaulay Matrix M_1

$$\begin{array}{c} \\ df_4 \\ f_3 \\ bf_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} ab & b^2 & bc & ad & bd & cd & d^2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Example: Cyclic-4

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Gaussian Elimination of M_1:

$$\begin{array}{c} \\ df_4 \\ f_3 \\ bf_4 \end{array}\begin{array}{c} \begin{array}{ccccccc} ab & b^2 & bc & ad & bd & cd & d^2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 1 \end{pmatrix} \end{array}$$

Example: Cyclic-4

Convert matrix data back to polynomial structure F_1:

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} ab & b^2 & bc & ad & bd & cd & d^2 \end{array} \\
\begin{array}{c} df_4 \\ f_3 \\ bf_4 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}
\end{array}
$$

$$
F_1 = \left\{ \underbrace{ad + bd + cd + d^2}_{f_5}, \underbrace{ab + bc - bd - d^2}_{f_6}, \underbrace{b^2 + 2bd + d^2}_{f_7} \right\}
$$

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$$F_1 = \left\{ \underbrace{ad + bd + cd + d^2}_{f_5}, \underbrace{ab + bc - bd - d^2}_{f_6}, \underbrace{b^2 + 2bd + d^2}_{f_7} \right\}$$

$\mathrm{lt}\left(f_5\right), \mathrm{lt}\left(f_6\right) \in L(G)$, so

$\mathbf{G} \leftarrow \mathbf{G} \cup \{\mathbf{f_7}\}$.

Example: Cyclic-4

Next round:

$$G = \{f_4, f_7\},\ P_2 = \{(f_2, bcf_4)\},\ L_2 = \{f_2, bcf_4\}.$$

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$$G = \{f_4, f_7\},\ P_2 = \{(f_2, bcf_4)\},\ L_2 = \{f_2, bcf_4\}.$$

We can simplify the computations:

$$\mathrm{lt}(bcf_4) = abc = \mathrm{lt}(cf_6).$$

f_6 possibly better reduced than f_4.(f_6 is not in G!)

$$\Longrightarrow L_2 = \{f_2, cf_6\}$$

Example: Cyclic-4

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$$\operatorname{lt}(bcf_4) = abc = \operatorname{lt}(cf_6).$$

f_6 possibly better reduced than f_4.(f_6 is not in G!)

$$\implies L_2 = \{f_2, cf_6\}$$

Symbolic preprocessing:

$$\begin{aligned} T(L_2) &= \{abc, bc^2, abd, acd, bcd, cd^2\} \\ L_2 &= \{f_2, cf_6, \quad\} \end{aligned}$$

Example: Cyclic-4

Next round:

$$G = \{f_4, f_7\},\ P_2 = \{(f_2, bcf_4)\},\ L_2 = \{f_2, bcf_4\}.$$

We can simplify the computations:

$$\mathrm{lt}(bcf_4) = abc = \mathrm{lt}(cf_6).$$

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f_6 possibly better reduced than f_4.(f_6 is not in G!)

$$\Longrightarrow\ L_2 = \{f_2, cf_6\}$$

Symbolic preprocessing:

$$\begin{aligned} T(L_2) &= \{abc, bc^2, abd, acd, bcd, cd^2\} \\ L_2 &= \{f_2, cf_6, \quad\} \end{aligned}$$

$bc^2 \notin L(G)$, $abd = \operatorname{lt}(bdf_4)$, but also $abd = \operatorname{lt}(bf_5)$!

Let us investigate this in more detail.

Interlude – Simplify

Idea

Try to replace $u \cdot f$ by a product $(wv) \cdot g$ where vg corresponds to an already computed row in the Gauss. Elim. of a previous matrix M_i.

$\Rightarrow$ Reuse rows that are reduced but not "in" G.

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Input: monomial u, polynomial f, list $\mathscr{F}$ of old F_i (from M_i after Gauss. Elim.)
Output: product $v \cdot g$ replacing $u \cdot f$

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Input: monomial u, polynomial f, list $\mathscr{F}$ of old F_i (from M_i after Gauss. Elim.)
Output: product $v \cdot g$ replacing $u \cdot f$

1. $d \leftarrow$ current index in the F4 algorithm
2. $D(u) \leftarrow \{$list of divisors of $u\}$
3. for $w \in D(u)$
   (a) if $\exists j \in \{1, \ldots, d-1\}$ such that $w \cdot f$ corresponds to row in M_j
      - $\triangleright$ $\exists_1\, g \in F_j$ such that $\operatorname{lt}(g) = \operatorname{lt}(w \cdot f)$
      - $\triangleright$ if $w \neq u$: Return **Simplify** $\left(\frac{u}{w}, g, \mathscr{F}\right)$ (recursive call)
      - $\triangleright$ else: Return $1 \cdot g$
4. else: Return $u \cdot f$

Interlude – Simplify

Note

- Tries to reuse all rows from old matrices.
  $\Rightarrow$ We need to keep them in memory.
- We also simplify generators of S-pairs, as we have done in our example: $(f_2, bcf_4) \Longrightarrow (f_2, cf_6)$.
- One can also choose “better” reducers by other properties, not only “last reduced one”.
- Without **Simplify** the F4 algorithm is rather slow.

Note

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- Without **Simplify** the F4 algorithm is rather slow.

In our example:
Choose bf_5 as reducer, not bdf_4.

Example: Cyclic-4

Symbolic preprocessing - now with simplify:

$$\begin{aligned} T(L_2) &= \{abc, bc^2, abd, acd, bcd, cd^2 \qquad\qquad\qquad\qquad \} \\ L_2 &= \{f_2, cf_6 \qquad\qquad\qquad \} \end{aligned}$$

$bc^2 \notin L(G)$,

Example: Cyclic-4

Symbolic preprocessing - now with simplify:

$$\begin{aligned} T(L_2) &= \{abc, bc^2, abd, acd, bcd, cd^2 \} \\ L_2 &= \{f_2, cf_6 \} \end{aligned}$$

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Example: Cyclic-4

Symbolic preprocessing - now with simplify:

$$\begin{aligned} T(L_2) &= \{abc, bc^2, abd, acd, bcd, cd^2, b^2d, c^2d \quad\} \\ L_2 &= \{f_2, cf_6, bf_5 \quad\} \end{aligned}$$

$bc^2 \notin L(G)$, $abd = \operatorname{lt}(bf_5)$,

Example: Cyclic-4

Symbolic preprocessing - now with simplify:

$$
\begin{aligned}
T(L_2) &= \{abc, bc^2, abd, acd, bcd, cd^2, b^2d, c^2d, \dots\} \\
L_2 &= \{f_2, cf_6, bf_5, cf_5, df_7\}
\end{aligned}
$$

$bc^2 \notin L(G)$, $abd = \mathrm{lt}(bf_5)$, and so on.

Example: Cyclic-4

Symbolic preprocessing - now with simplify:

$$\begin{aligned} T(L_2) &= \{abc, bc^2, abd, acd, bcd, cd^2, b^2d, c^2d, \ldots\} \\ L_2 &= \{f_2, cf_6, bf_5, cf_5, df_7\} \end{aligned}$$

$bc^2 \notin L(G)$, $abd = \mathrm{lt}(bf_5)$, and so on.

Now try to exploit the special structure of the Macaulay matrices.

Improve Gaussian Elimination

Use **Linear Algebra** for reduction steps in GB computations.

Improve Gaussian Elimination

Use **Linear Algebra** for reduction steps in GB computations.

$$\begin{matrix} 1 & 3 & 0 & 0 & 7 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 5 \\ 0 & 1 & 6 & 0 & 8 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 1 \end{matrix}$$

Improve Gaussian Elimination

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$$\begin{matrix} 1 & 3 & 0 & 0 & 7 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 5 \\ 0 & 1 & 6 & 0 & 8 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 1 \end{matrix}$$

Knowledge of underlying GB structure

Improve Gaussian Elimination

Use **Linear Algebra** for reduction steps in GB computations.

$$
\begin{array}{rc}
\text{S-pair} & \left\{ \begin{array}{ccccccc} 1 & 3 & 0 & 0 & 7 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 5 \end{array} \right. \\
\text{S-pair} & \left\{ \begin{array}{ccccccc} 0 & 1 & 6 & 0 & 8 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \end{array} \right. \\
\text{reducer} \quad \leftarrow & \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 1 & 3 & 1 \end{array}
\end{array}
$$

Knowledge of underlying GB structure

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$$

Knowledge of underlying GB structure

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\text{S-pair} & \left\{\begin{array}{ccccccc} 1&3&0&0&7&1&0\\ 1&0&4&1&0&0&5 \end{array}\right.\\
\text{S-pair} & \left\{\begin{array}{ccccccc} 0&1&6&0&8&0&1\\ 0&5&0&0&0&2&0 \end{array}\right.\\
\text{reducer} \quad \leftarrow & \;\;\begin{array}{ccccccc} 0&0&0&0&1&3&1 \end{array}
\end{array}$$

Knowledge of underlying GB structure

Idea

Do a static **reordering before** the Gaussian Elimination to achieve a better initial shape. **Reorder afterwards**.

Faugère-Lachartre Idea

1st step: Sort pivot and non-pivot columns

$$\begin{matrix} 1 & 3 & 0 & 0 & 7 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 5 \\ 0 & 1 & 6 & 0 & 8 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 1 \end{matrix}$$

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Pivot column

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0 & 1 & 6 & 0 & 8 & 0 & 1 \\
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\end{matrix}
$$

Pivot column

Non-Pivot column

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1 & 3 & 0 & 0 & 7 & 1 & 0 \\
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0 & 0 & 0 & 0 & 1 & 3 & 1
\end{array}
\longrightarrow
\begin{array}{ccc|cccc}
1 & 3 & 7 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
0 & 1 & 8 & 6 & 0 & 0 & 9 \\
0 & 5 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 3 & 1
\end{array}
$$

Pivot column

Non-Pivot column

Faugère-Lachartre Idea

2nd step: Sort pivot and non-pivot rows

$$\begin{array}{ccc:cccc}
1 & 3 & 7 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
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\begin{array}{ccc:cccc}
1 & 3 & 7 & 0 & 0 & 1 & 0 \\
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0 & 0 & 1 & 0 & 0 & 3 & 1
\end{array}
$$

Pivot row

Faugère-Lachartre Idea

2nd step: Sort pivot and non-pivot rows

$$
\begin{array}{ccc:cccc}
1 & 3 & 7 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
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\end{array}
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Pivot row

Non-Pivot row

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\begin{array}{ccc:cccc}
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Pivot row

Non-Pivot row

Faugère-Lachartre Idea

3rd step: Reduce lower left part to zero

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 4 & 1 & 0 & 5 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 \\ \hline 1 & 3 & 7 & 0 & 0 & 1 & 0 \\ 0 & 1 & 8 & 6 & 0 & 0 & 9 \end{array}\right)$$

Faugère-Lachartre Idea

3rd step: Reduce lower left part to zero

$$\begin{array}{ccc|cccc} 1 & 0 & 0 & 4 & 1 & 0 & 5 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 \\ \hline 1 & 3 & 7 & 0 & 0 & 1 & 0 \\ 0 & 1 & 8 & 6 & 0 & 0 & 9 \end{array} \longrightarrow \begin{array}{ccc|cccc} 1 & 0 & 0 & 4 & 1 & 0 & 5 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 \\ \hline 0 & 0 & 0 & 7 & 10 & 3 & 10 \\ 0 & 0 & 0 & 6 & 0 & 2 & 1 \end{array}$$

Faugère-Lachartre Idea

4th step: Reduce lower right part

$$
\left(\begin{array}{ccc:cccc}
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
0 & 5 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 3 & 1 \\
\hdashline
0 & 0 & 0 & 7 & 10 & 3 & 10 \\
0 & 0 & 0 & 6 & 0 & 2 & 1
\end{array}\right)
$$

Faugère-Lachartre Idea

4th step: Reduce lower right part

$$\left(\begin{array}{ccc:cccc} 1 & 0 & 0 & 4 & 1 & 0 & 5 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 \\ \hdashline 0 & 0 & 0 & 7 & 10 & 3 & 10 \\ 0 & 0 & 0 & 6 & 0 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc:cccc} 1 & 0 & 0 & 4 & 1 & 0 & 5 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 \\ \hdashline 0 & 0 & 0 & 7 & 10 & 3 & 10 \\ 0 & 0 & 0 & 0 & 4 & 1 & 5 \end{array}\right)$$

Faugère-Lachartre Idea

4th step: Reduce lower right part

$$
\left(\begin{array}{ccc:cccc}
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
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\hdashline
0 & 0 & 0 & 7 & 10 & 3 & 10 \\
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1 & 0 & 0 & 4 & 1 & 0 & 5 \\
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\hdashline
0 & 0 & 0 & 7 & 10 & 3 & 10 \\
0 & 0 & 0 & 0 & 4 & 1 & 5
\end{array}\right)
$$

5th step: Remap columns of lower right part

How our matrices look like

Some data about the matrix:

- F_4 computation of homogeneous KATSURA-12, degree 6 matrix

How our matrices look like

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How our matrices look like

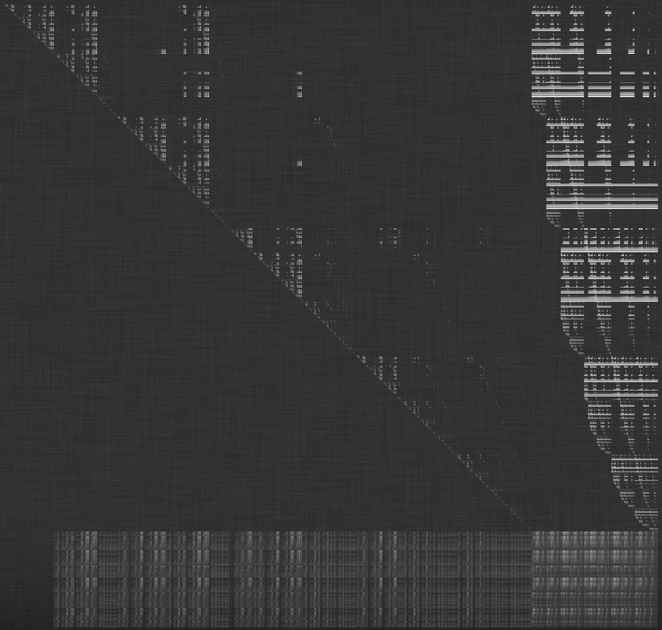
Some data about the matrix:

- F_4 computation of homogeneous KATSURA-12, degree 6 matrix
- Size $137MB$
- $24,006,869$ nonzero elements (density: 5%)
- Dimensions:

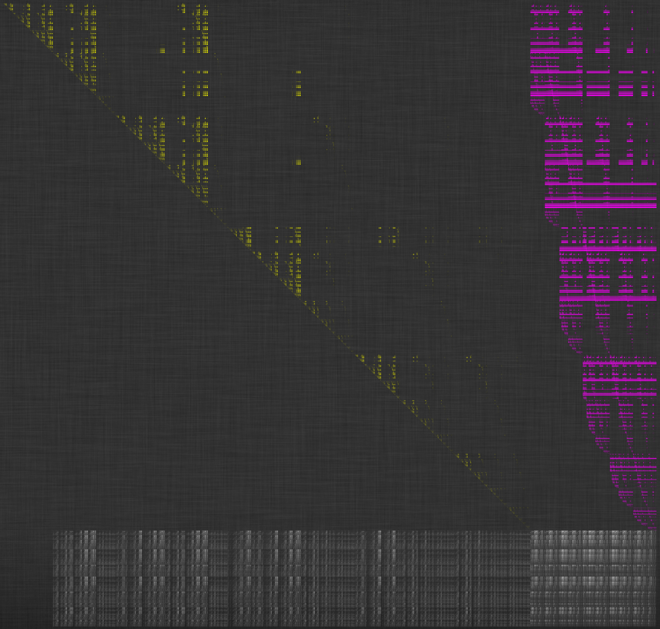
full matrix:	$21,182$	$\times$	$22,207$
upper-left:	$17,915$	$\times$	$17,915$
lower-left:	$3,267$	$\times$	$17,915$
upper-right:	$17,915$	$\times$	$4,292$
lower-right:	$3,267$	$\times$	$4,292$

How our matrices look like (2)

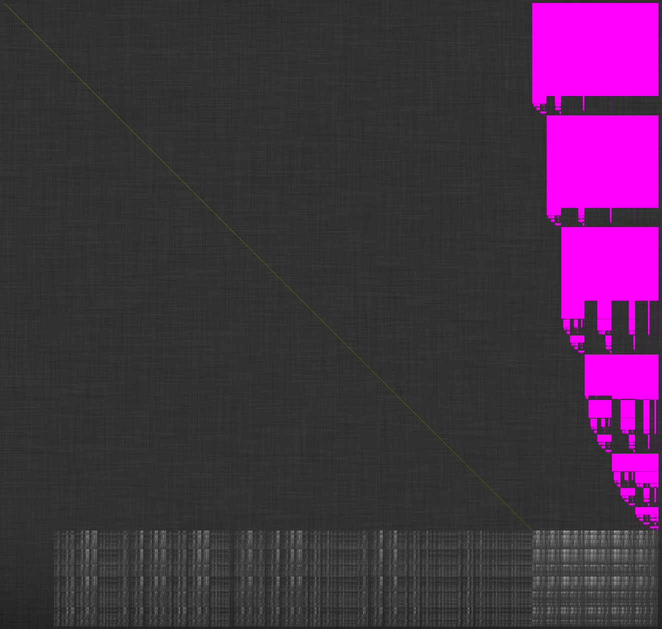
How our matrices look like (3)



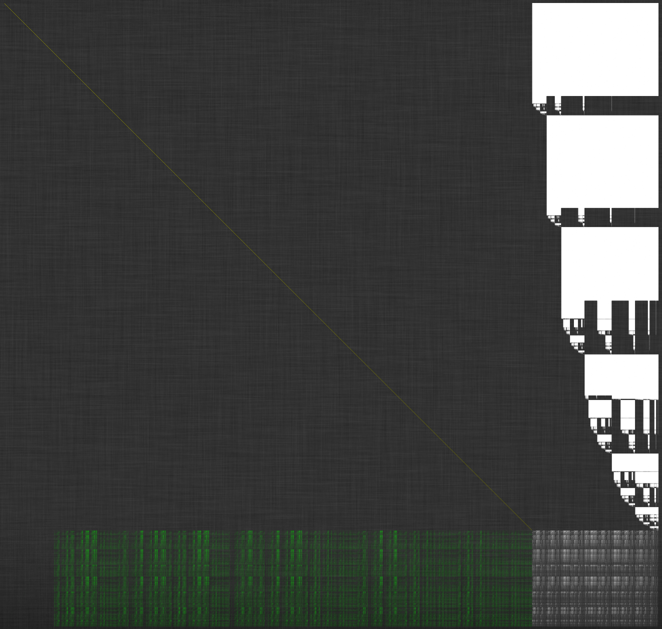
Hybrid Matrix Multiplication $A^{-1}B$



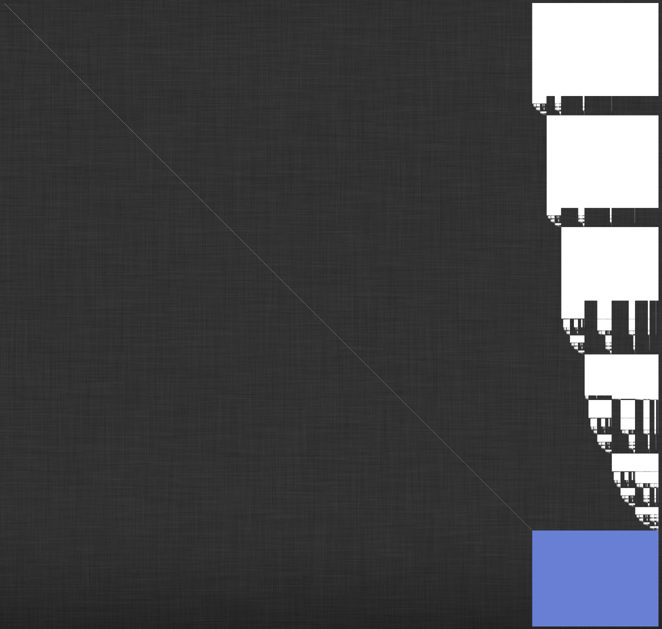
Hybrid Matrix Multiplication $A^{-1}B$



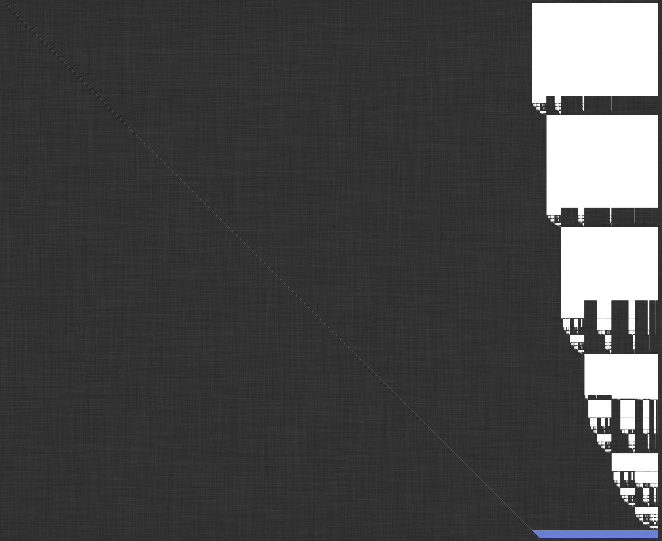
Reduce C to zero



Gaussian Elimination on D



New information



First attempts

2010 – UPMC Paris 6, INRIA PolSys Team
Jean-Charles Faugère & Sylvain Lachartre – FL

2011 – University of Kaiserslautern
Bradford Hovinen – LELA
`https://github.com/Singular/LELA`

2012 – UPMC Paris 6, INRIA PolSys Team
Fayssal Martani – new implementation in LELA
`https://github.com/martani/LELA`

2012-2013 – University of Kaiserslautern
Bjarke Hammersholt Roune – MathicGB
`https://github.com/broune/mathicgb`

2012-2014 – University of Passau
Severin Neumann – parallelGBC
`https://github.com/svrnm/parallelGBC`

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