The effect of noise on the number of extreme points

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Introduction

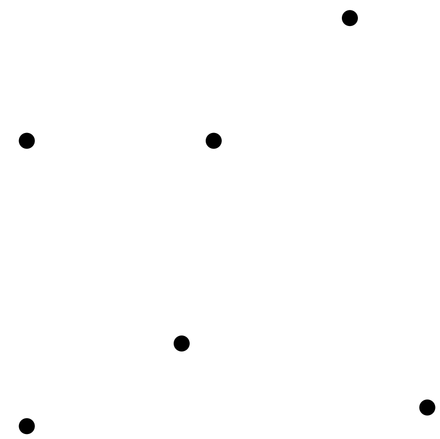
Geometric data structure

Complexity analysis

Limits: structure and precision

Geometric data structures

Let P be a set of n points...



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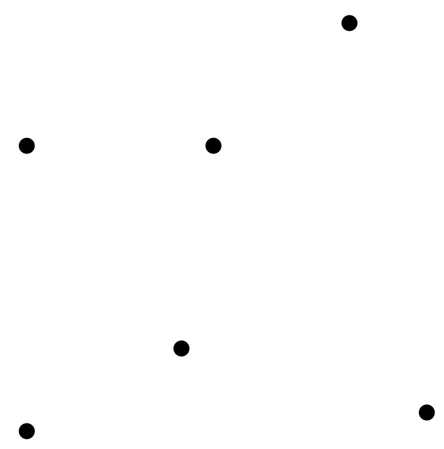

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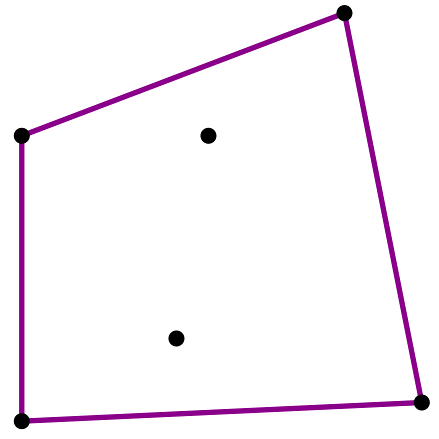

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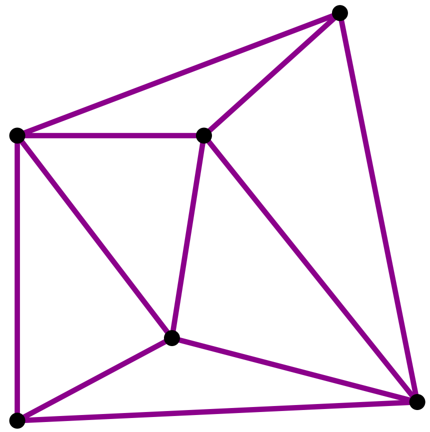

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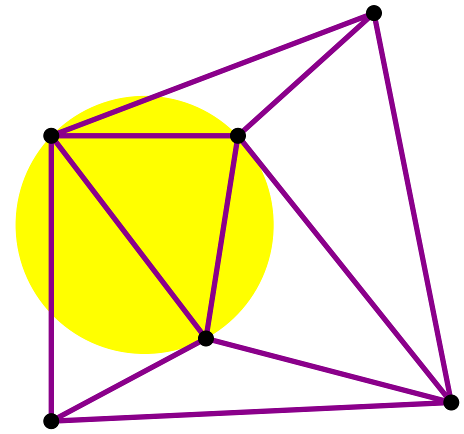

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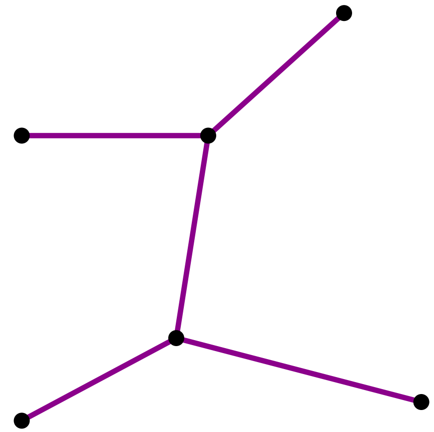

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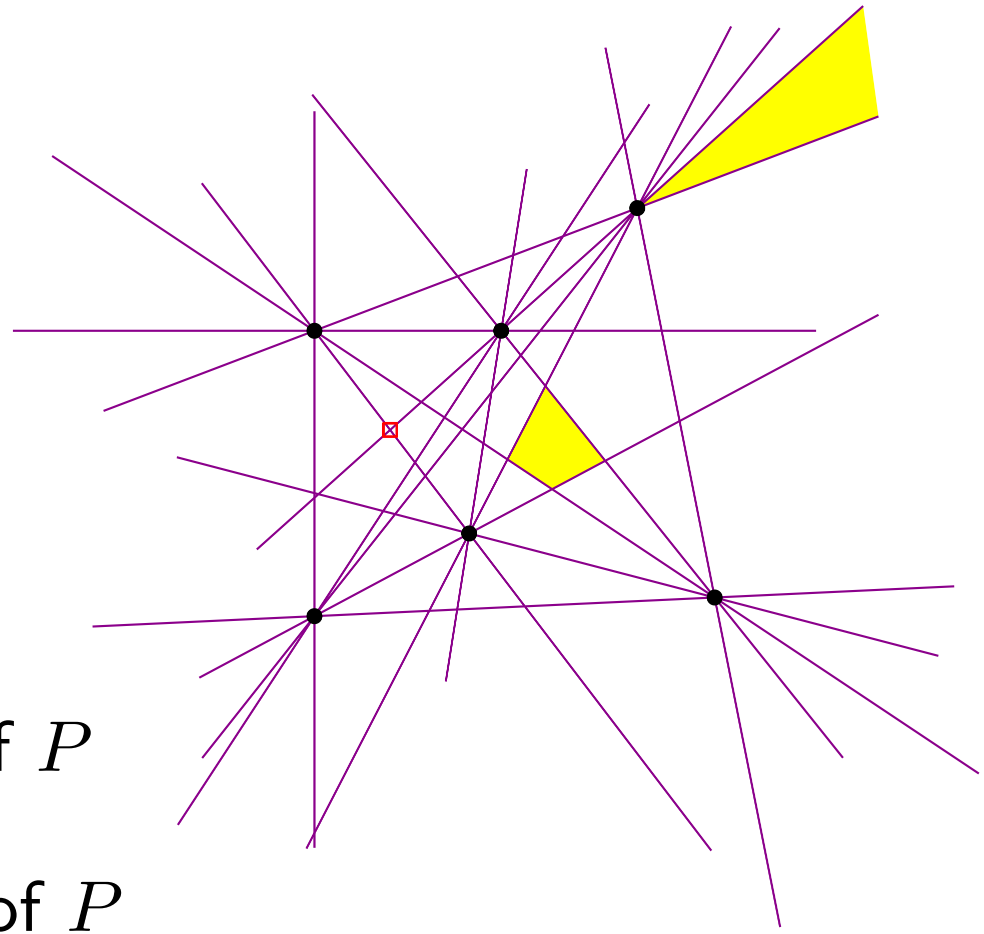

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- $\star$ the *arrangement* of the lines spanned by P

Complexity analysis

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Convex hull: $\Theta\left(n^{\lfloor d/2 \rfloor}\right)$

Delaunay triangulation: $\Theta\left(n^{\lceil d/2 \rceil}\right)$

Minimum spanning tree: $n-1$

Arrangement of induced hyperplanes: $\Theta\left(n^{d^2}\right)$

First limit: structure

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Still no good model of "computer graphic scene"...

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Spread-sensitive bounds
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"*Nice*" *sample* of a "nice" surface

[GN 02], [GN03], [ABL 03], [DEG 08], [AAD 09]

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Size of DT $\leftrightarrow$ "dimension" of the point set.

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Input points given in finite-precision arithmetic.

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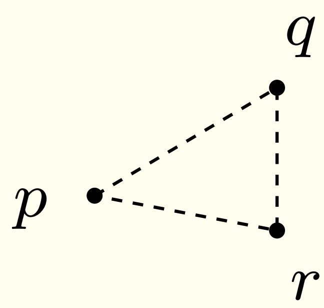

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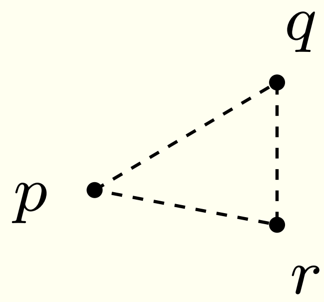

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Data is often noisy, lower-bounds are often carefully designed.

Problems

Geometric structures defined by finite-precision data?

Robustness of lower/upper bounds to noise?

Smoothed complexity analysis

General principles

Geometric probabilities

Shape matters

Smoothed complexity

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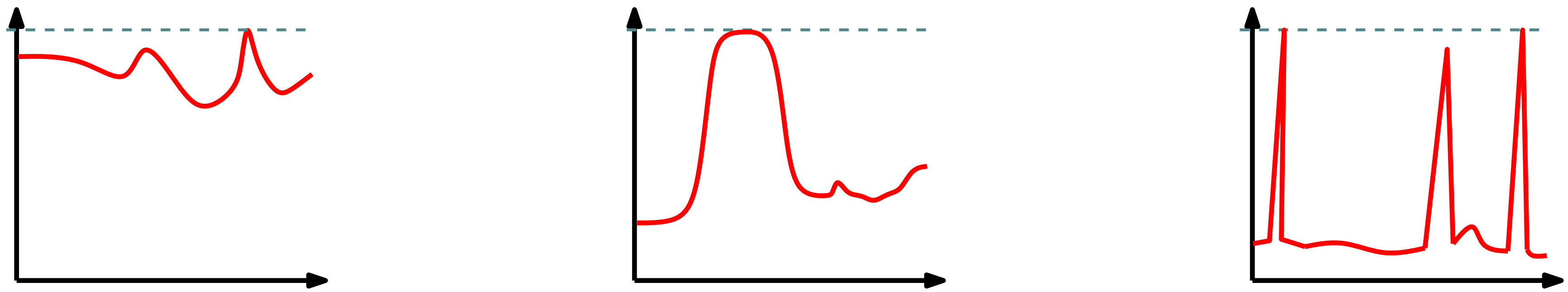

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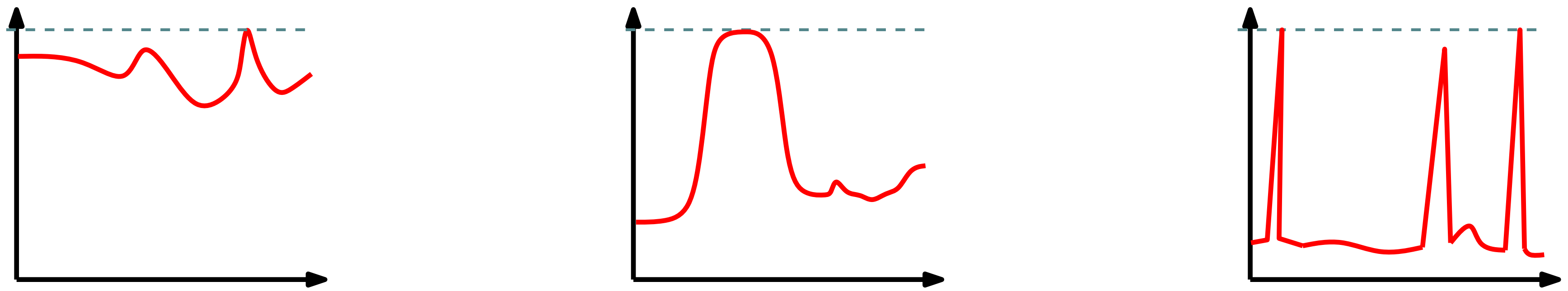

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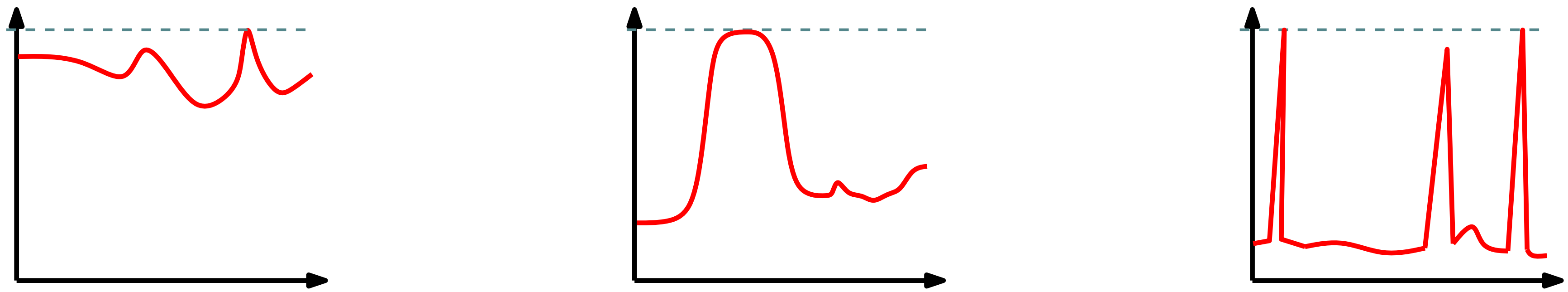

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$\simeq$ Convolution with a distribution concentrated near the origin.

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Gödel prize in 2008

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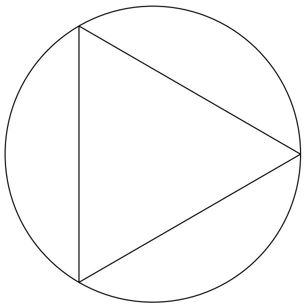

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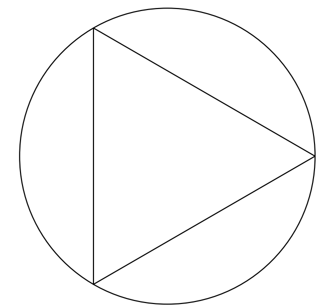

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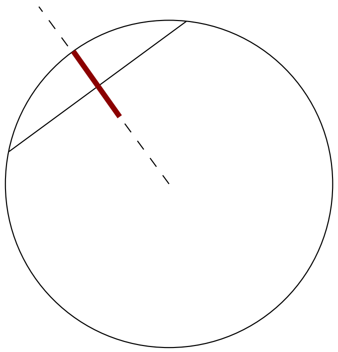

$1/2$

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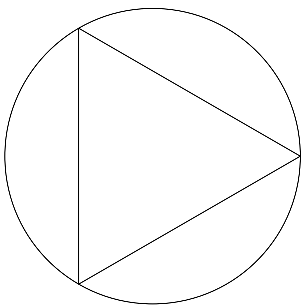

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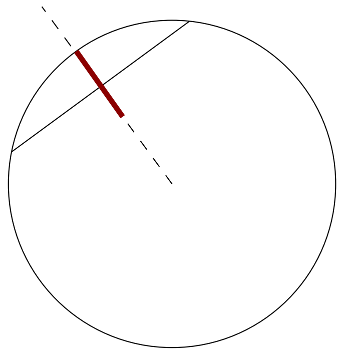

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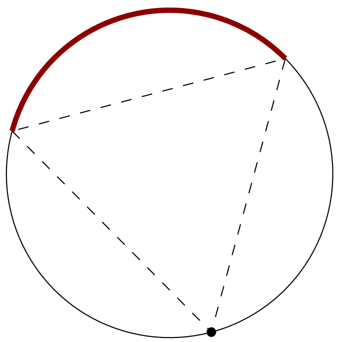

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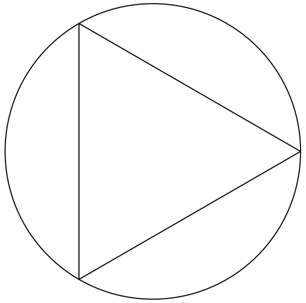

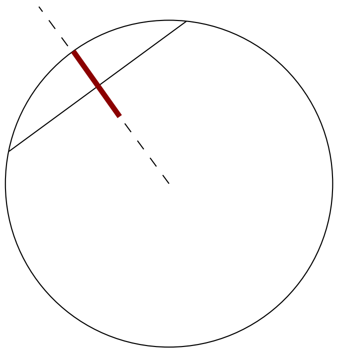

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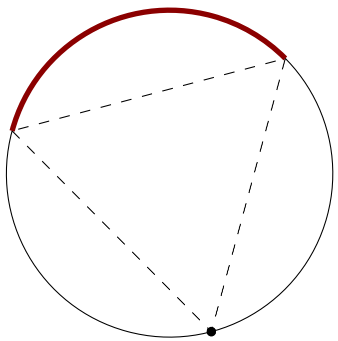

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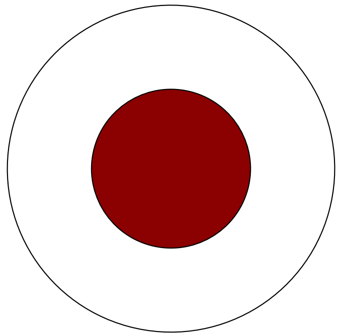

$1/4$

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Intuition: points chosen "close to corners" dominate many points.

Number of extreme points

The *shuffled convex hull* game

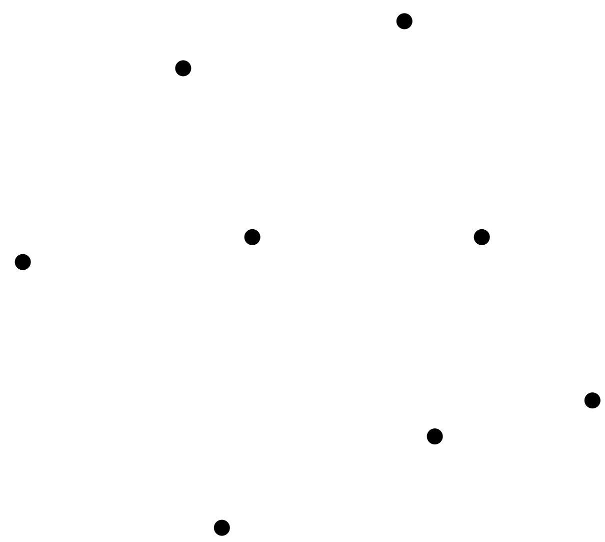
Our results

Comparison to "experimental" data

Shuffled convex hull game

Let X be a set of n points in some fixed domain D.

Y a L^p perturbation of X of amplitude δ.



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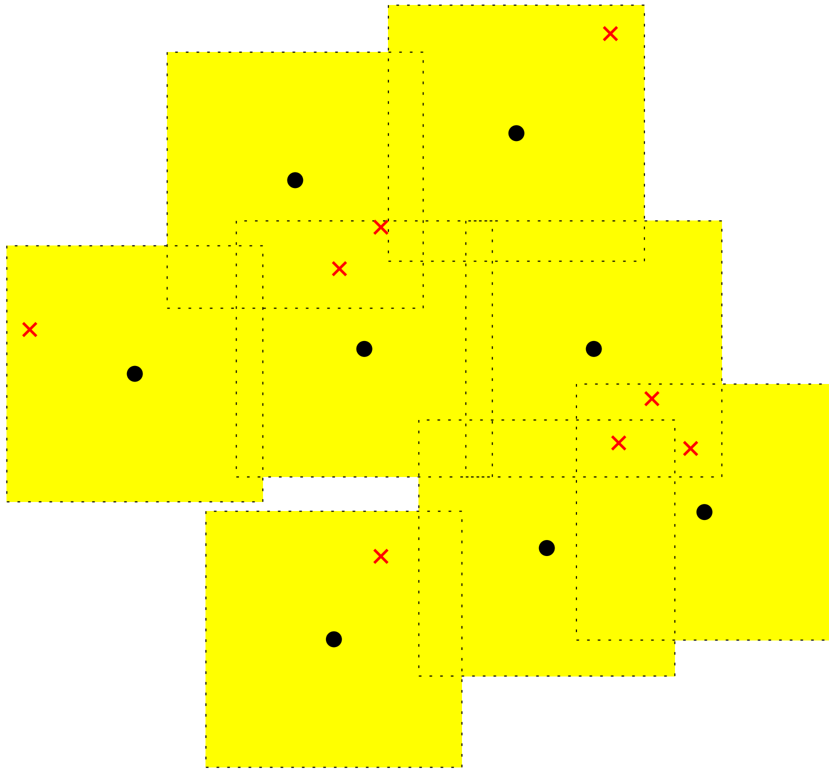

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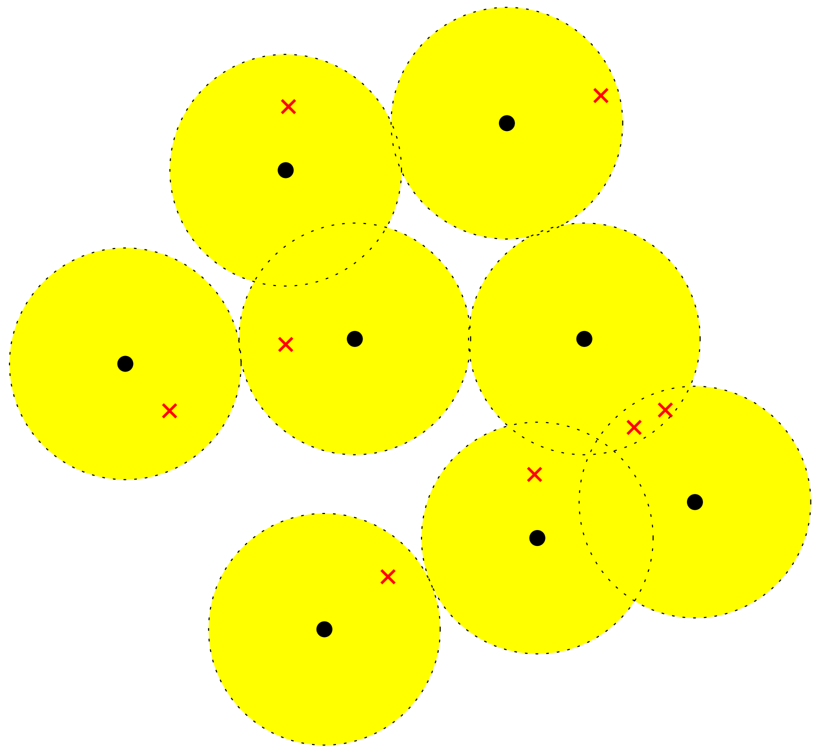

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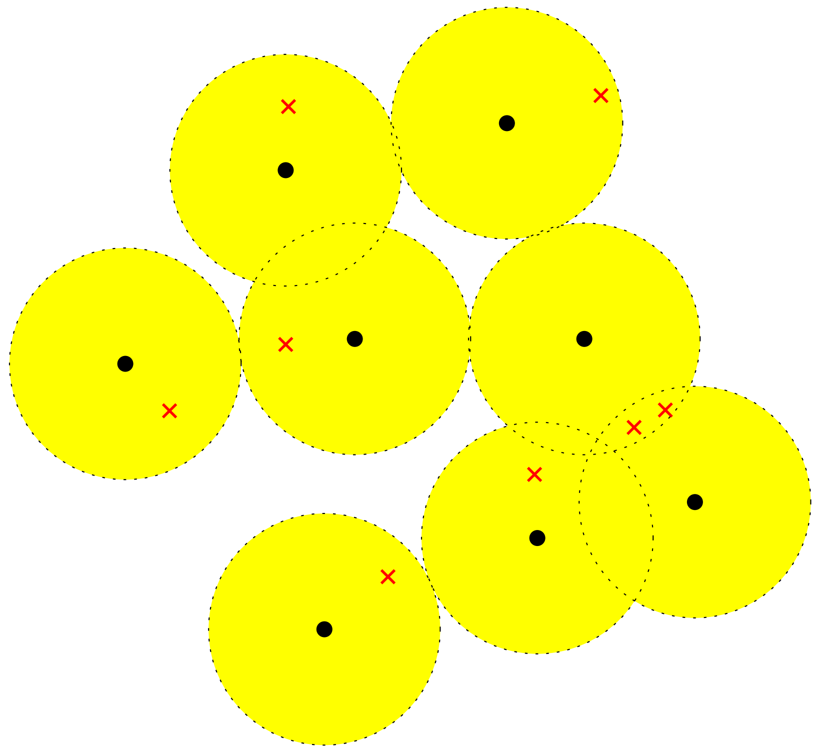
Given ϵ and p, find X such that $E[\#CH(Y)]$ is max.



Different D, different answers?

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perturbation	$E[\#CH(Y)]$	range of δ
L^1, L^∞	$\tilde{\Theta}\left(n^{1/5}\delta^{-2/5}\right)$	$\delta \in \left(\tilde{\Omega}(1/n^2), O(1)\right)$
L^2	$\tilde{\Theta}\left(n^{1/4}\delta^{-3/8}\right)$	

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Uniform noise simulated by pseudo-random number generators.

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regular n-gon → perturbed point set

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$\delta = 10^j$ for $j = -7, \ldots, 5$

average over 1000 - 100 trials

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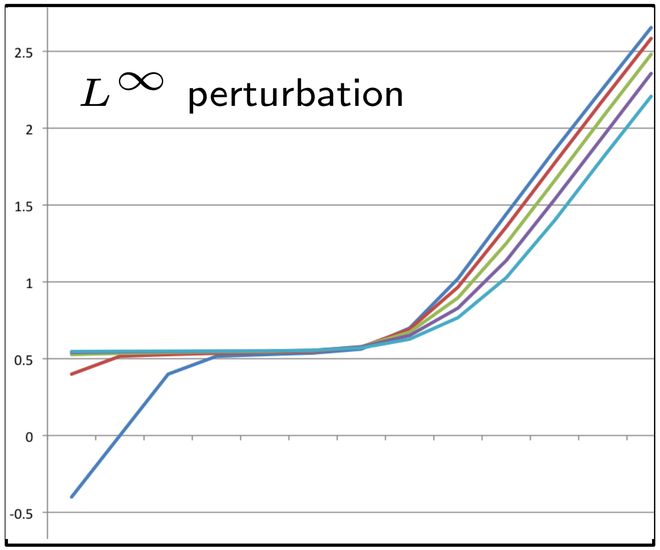
Plot $\log_{10} \#CH(Y) - \log_{10}(n^{1/5}\delta^{-2/5})$



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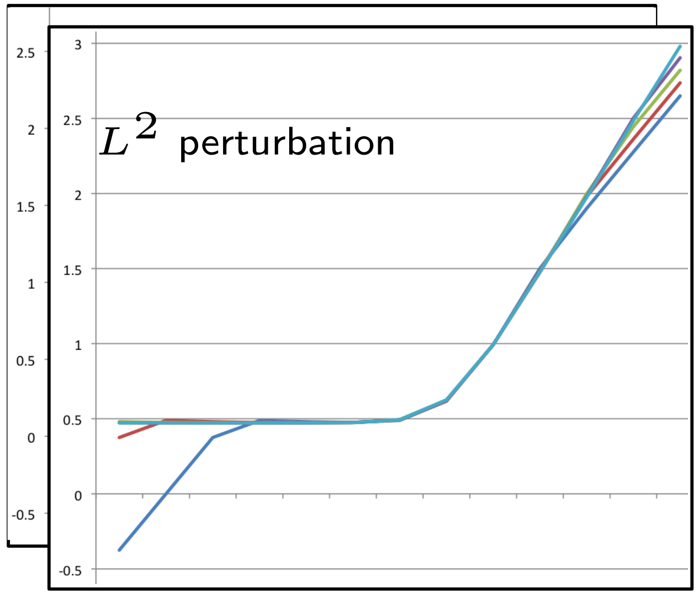

Comparison to exp. data

Effect of rounding coordinates to a coarse grid.

regular n-gon ⟶ rounded point set

1. rounding to the *double* grid
2. rounding to the *float* grid.

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average over 1000 - 100 trials

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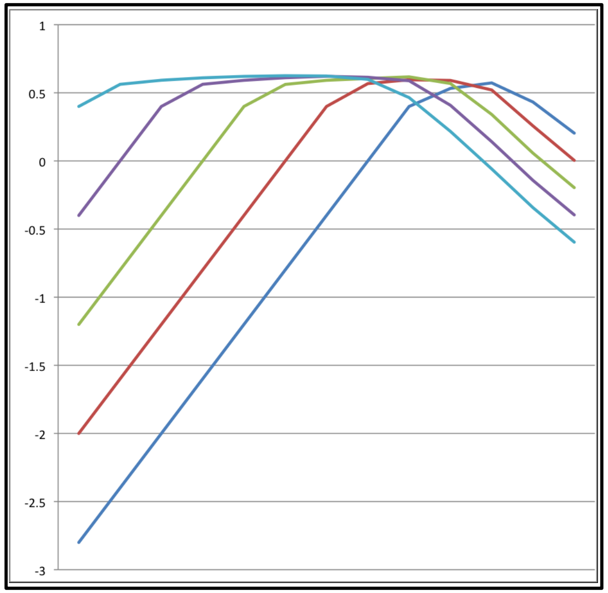


To summarize

Near-tight bounds for L^2 and L^∞.

Predicts the behavior of regular rounding quite accurately.

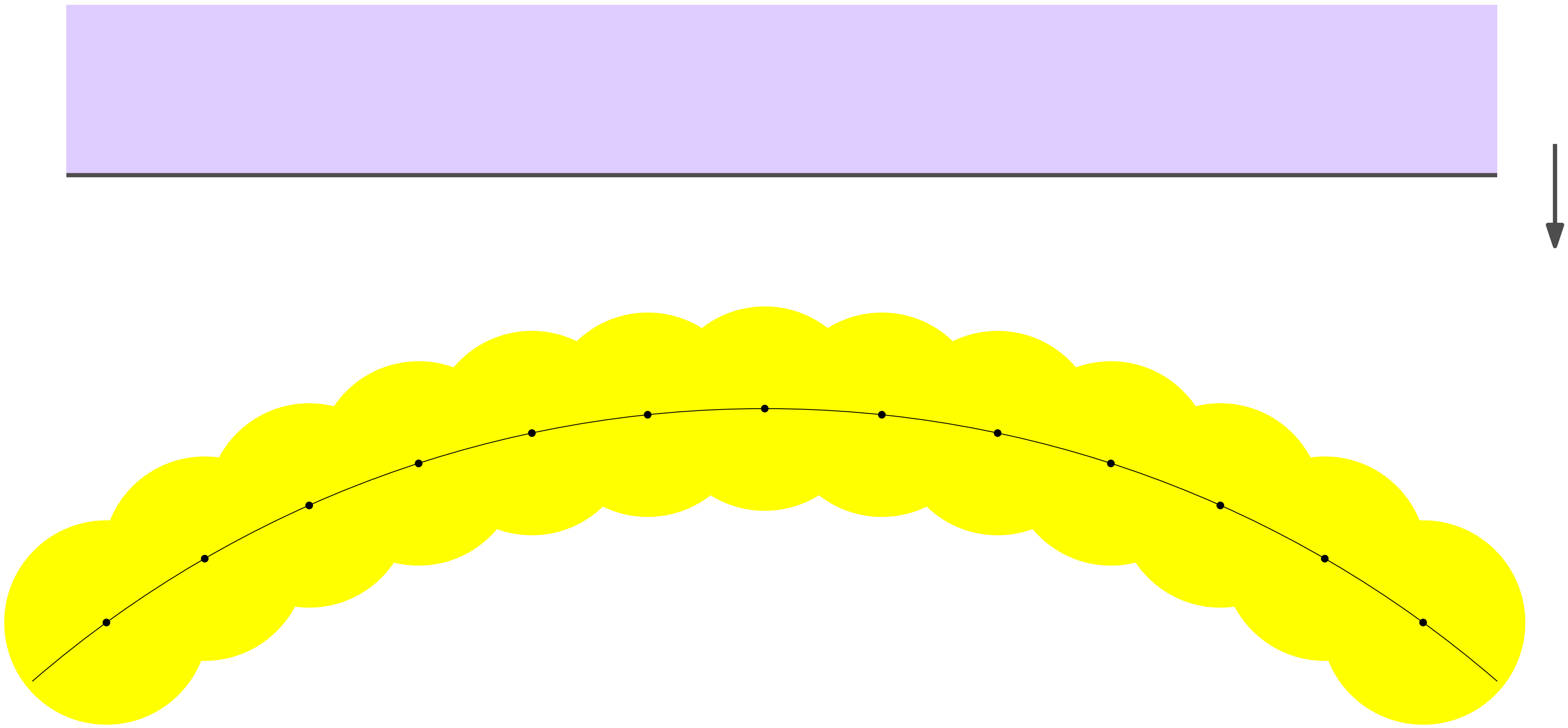
Generalizes to arbitrary dimension for L^2 perturbations.

A look under the hood

Witnesses and collectors

Shape matters (bis)

Principle



Push a halfplane and count the average # of points it contains.

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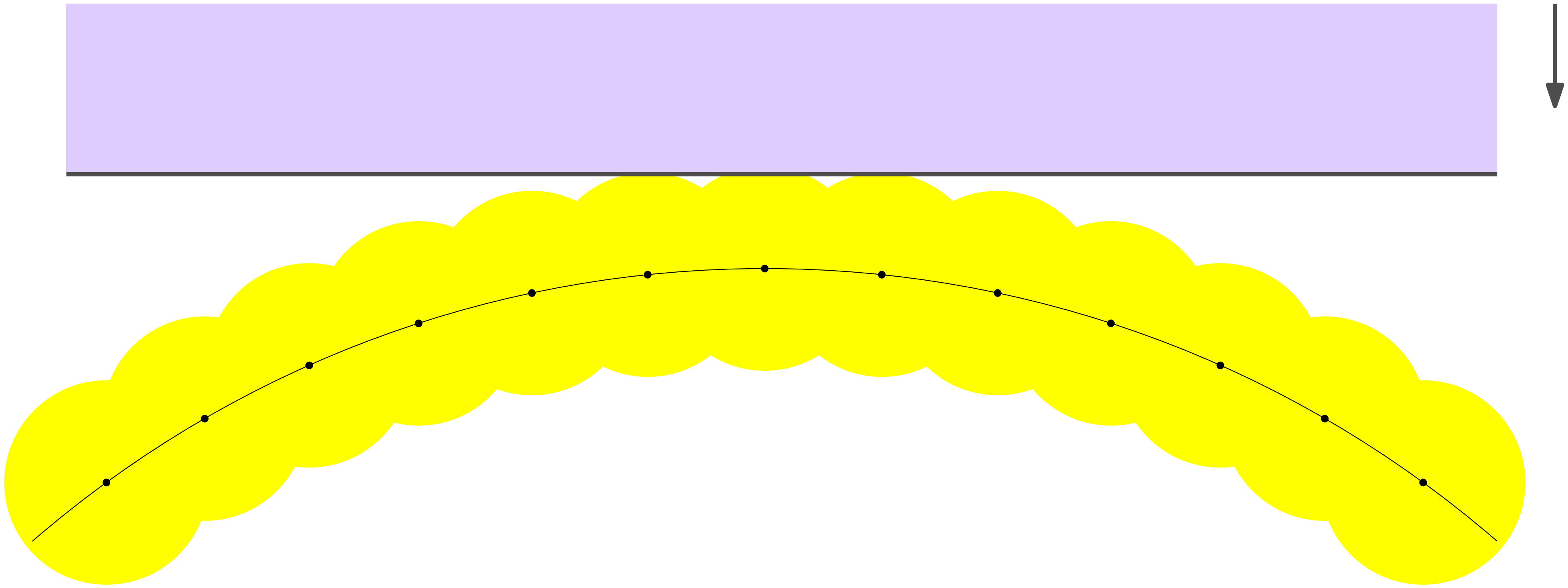

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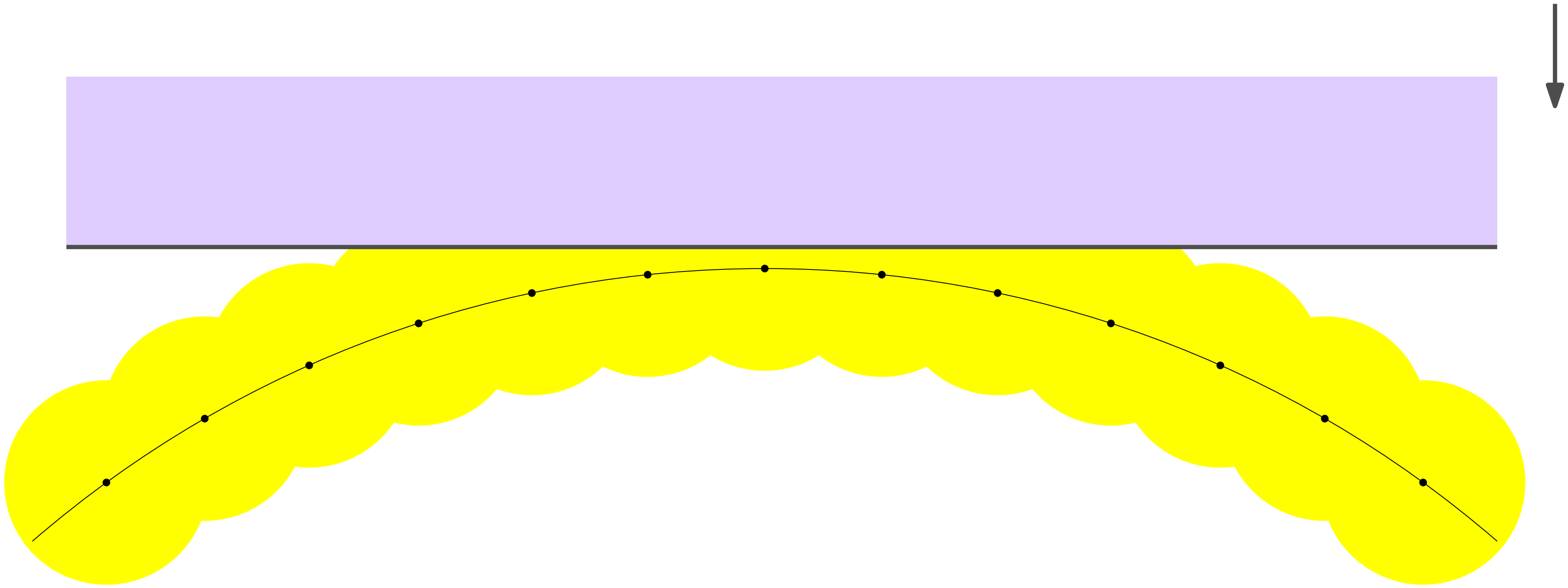

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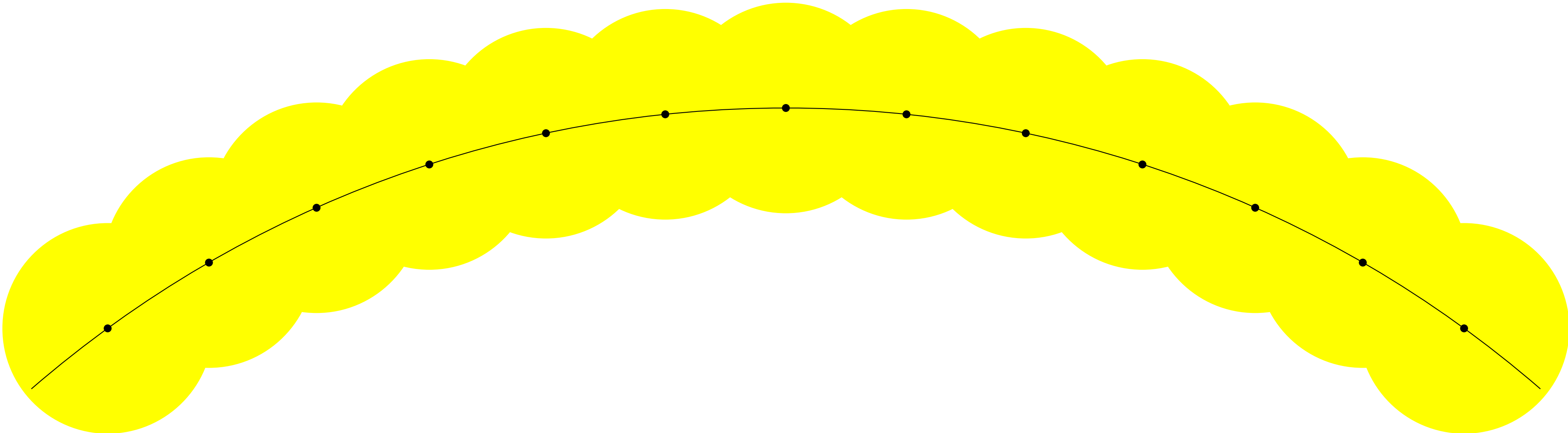

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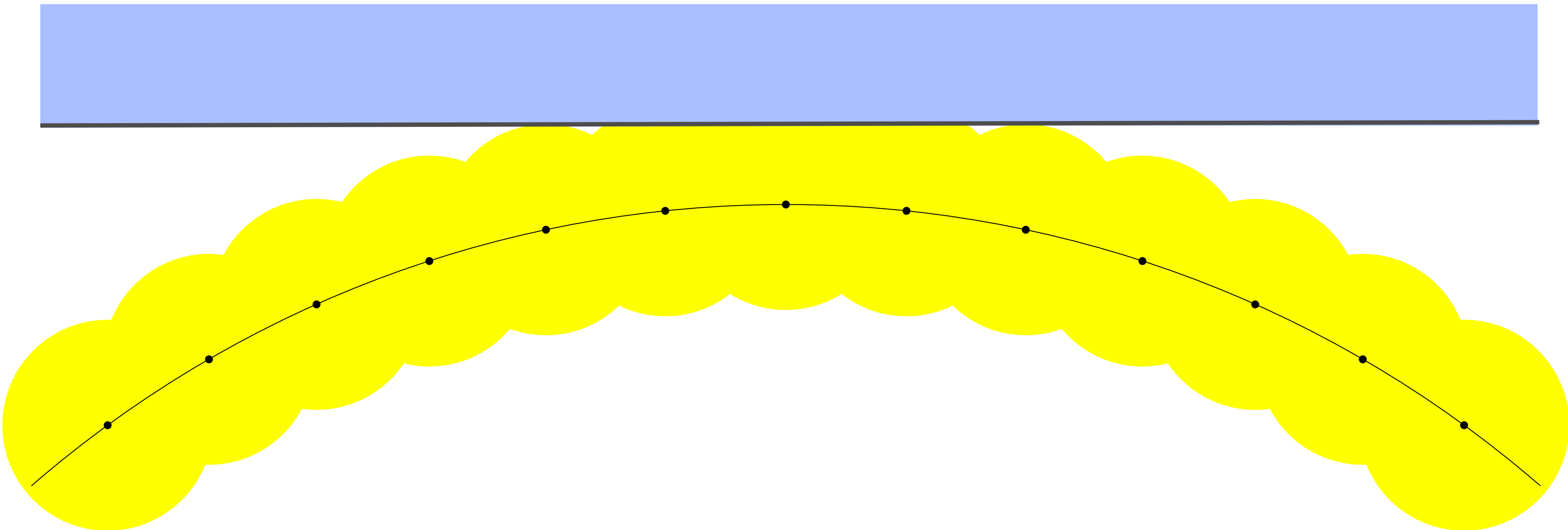

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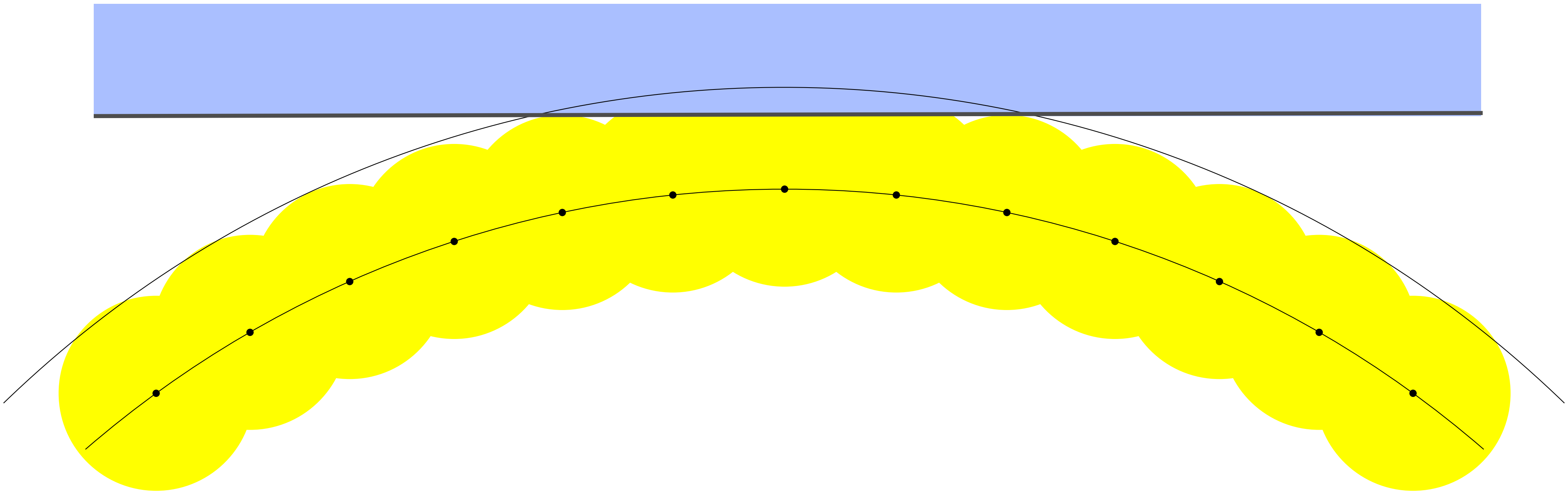

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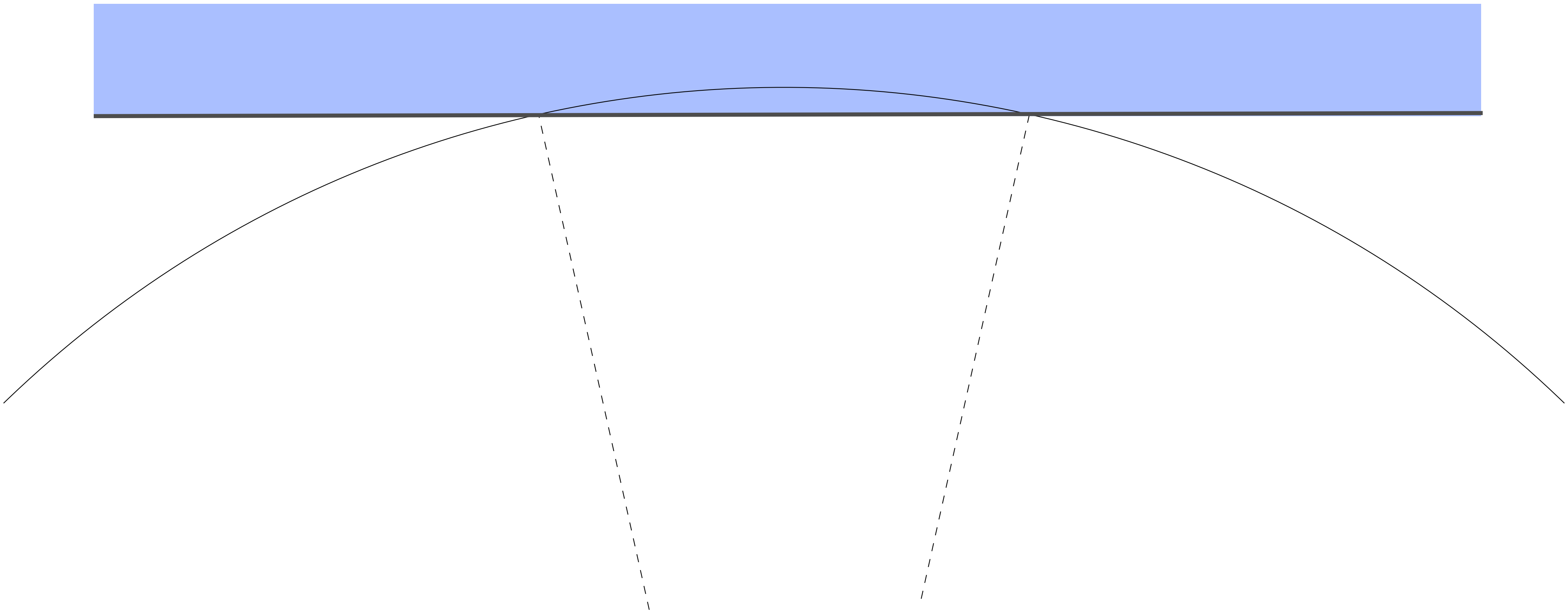

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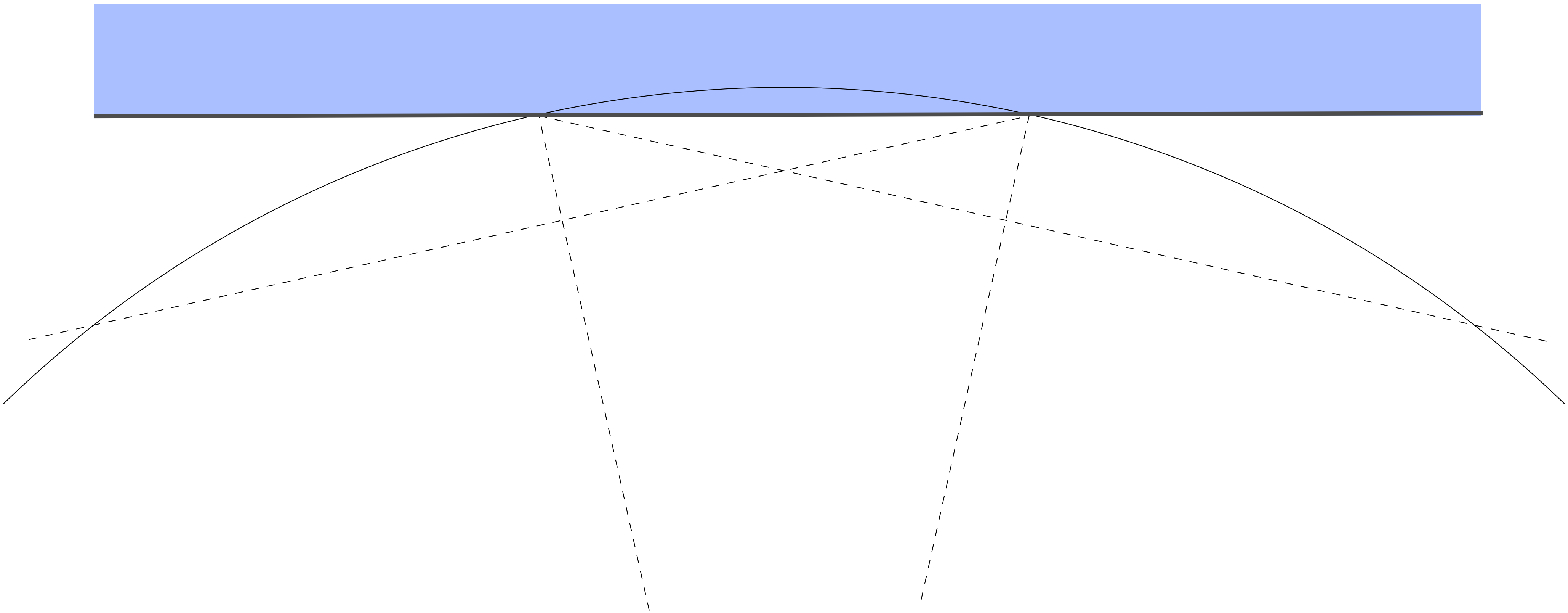

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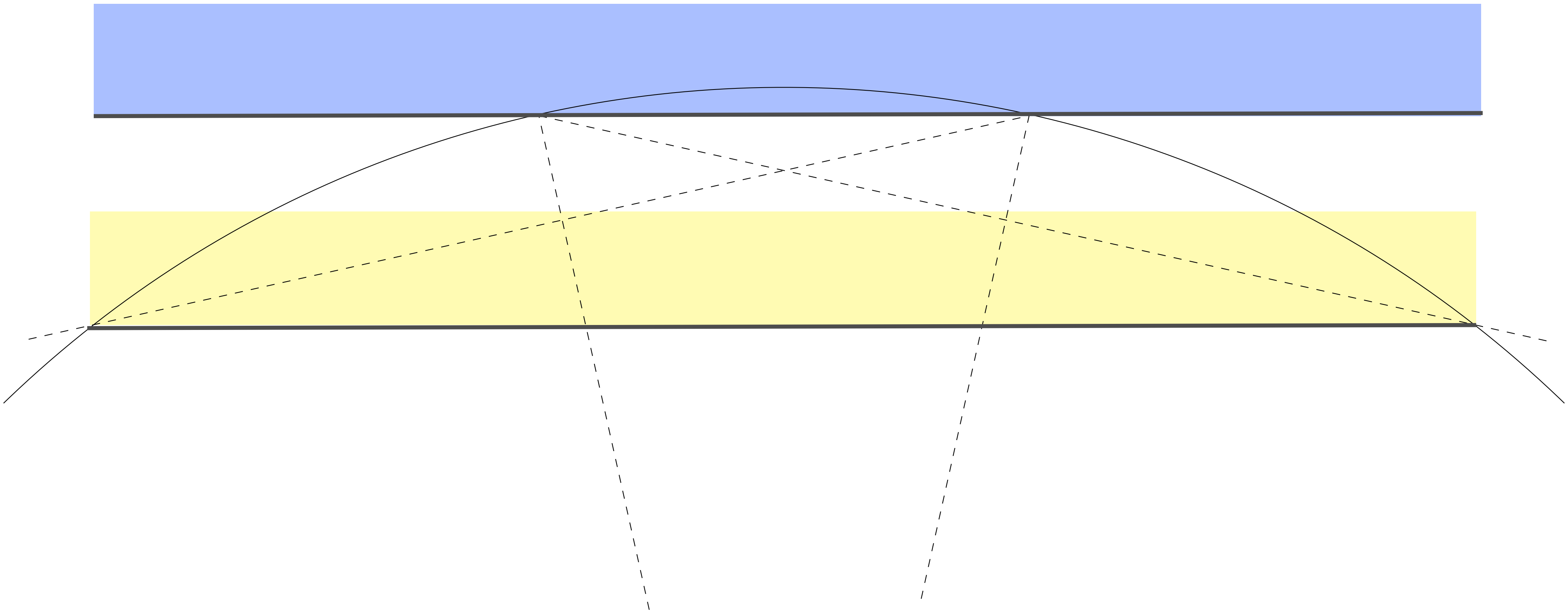

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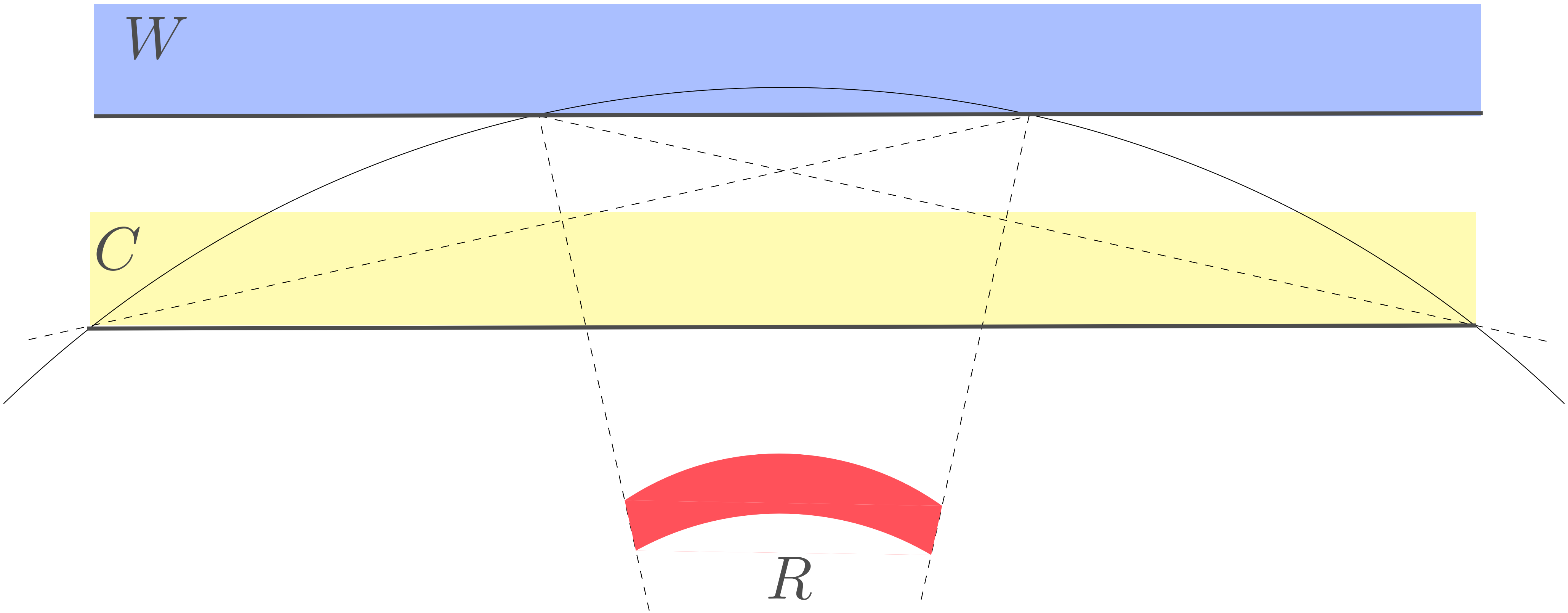

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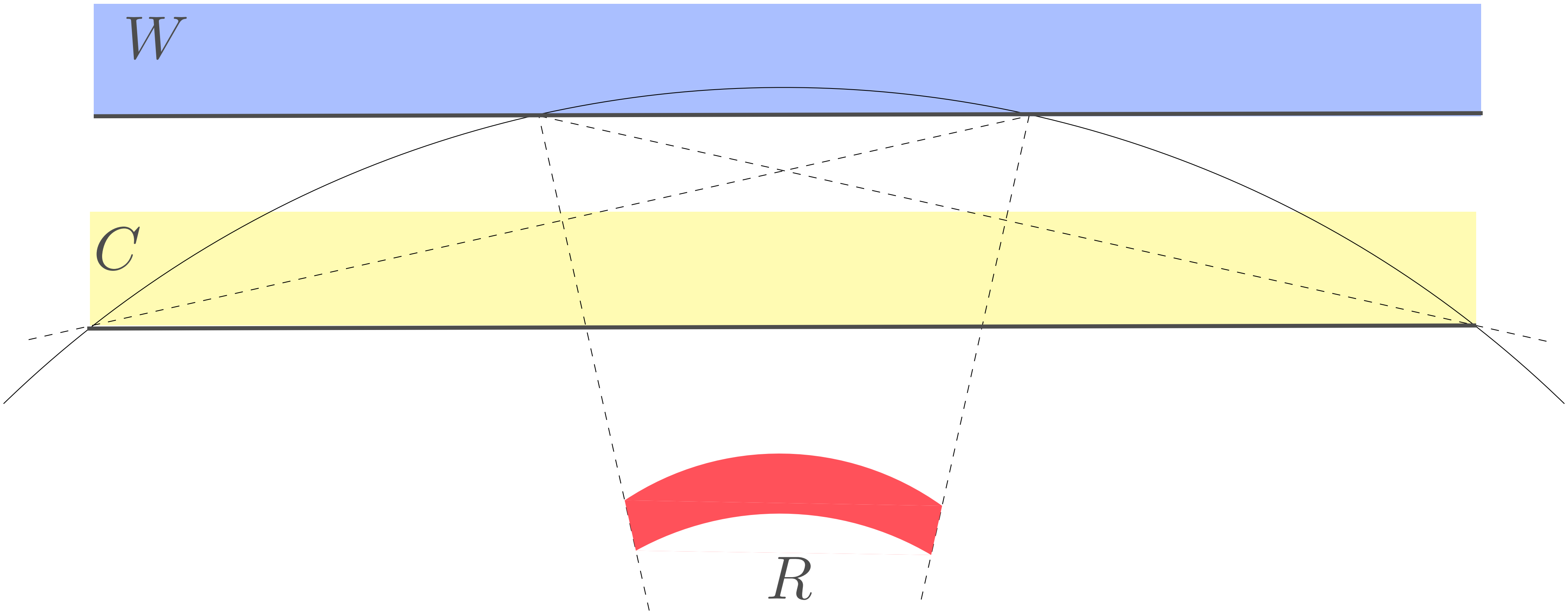

Any point in W dominates any point outside C for any direction in R.

Witnesses & collectors

Place enough pairs (C, W) so that the R's cover $\mathbb{S}^1$...

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$$\Omega\left(\Sigma(1 - e^{-E[W_i \cap Y]})\right) \leq E[\#CH(Y)]$$

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$$E[\#CH(Y)] \leq n e^{-\Sigma E[W_i \cap Y]} + \Sigma E[C_i \cap Y]$$

... while having any perturbation disk meet $O(1)$ W's.

$$\Omega\left(\Sigma(1 - e^{-E[W_i \cap Y]})\right) \leq E[\#CH(Y)]$$

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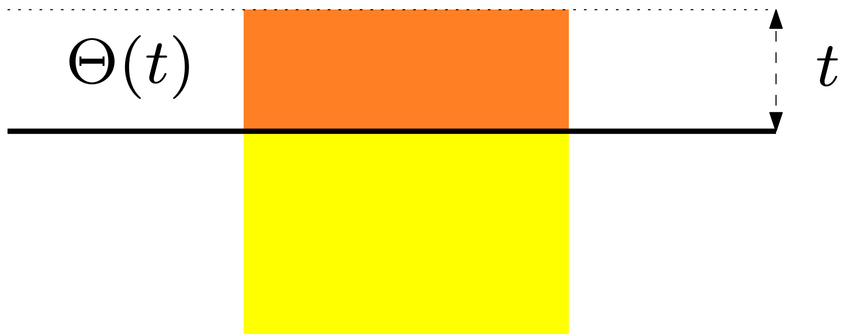
$$\Rightarrow E[\#CH(Y)] = \Theta\left(\#\text{ pairs }(C, W)\right)$$

Shape matters! (bis)



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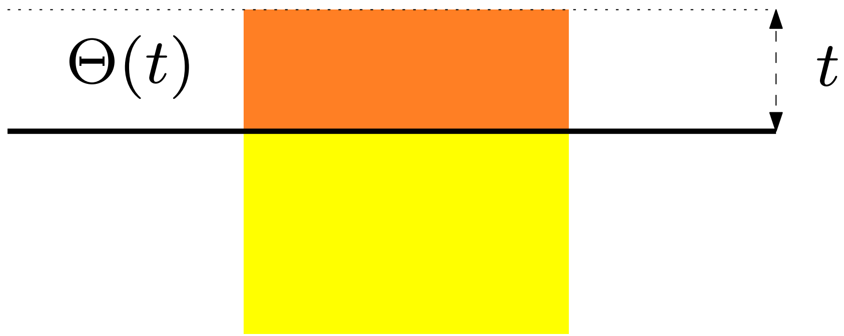

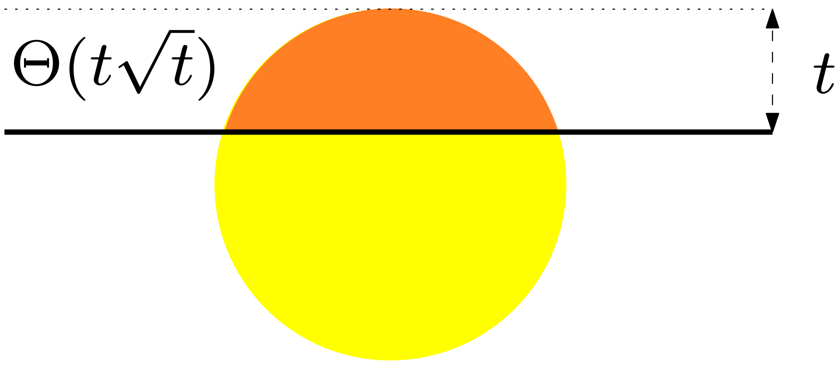

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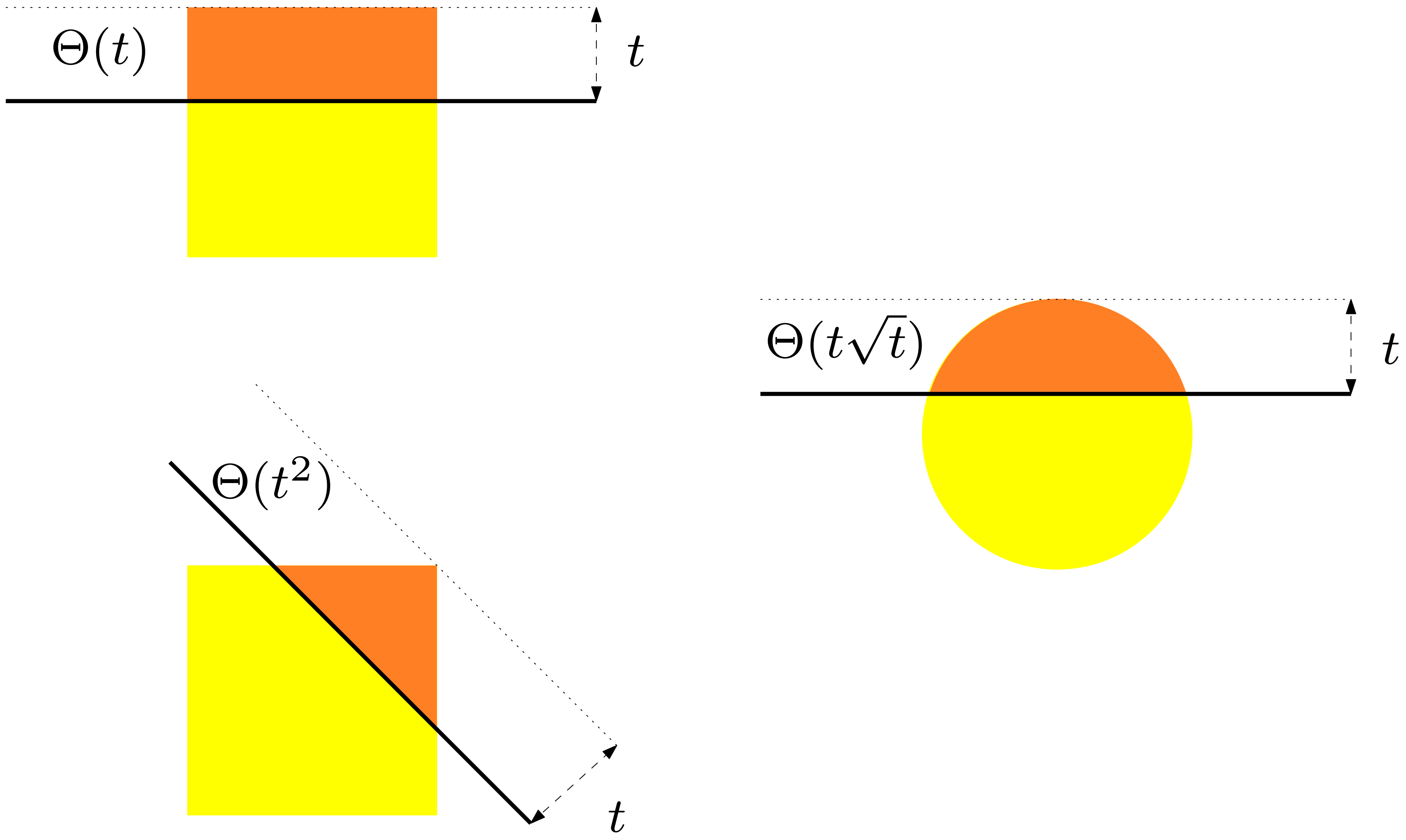

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$\square$