Towards a New Subexponential Factoring Algorithm

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Outline

Motivation

- Understanding factorisation and especially why the Number Field Sieve is the best current factoring approach.

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- Understanding factorisation and especially why the Number Field Sieve is the best current factoring approach.
- Understand why a more "natural " approach using the Riemann ζ function fails so far. Are we doomed to bang into a wall through an *analytic* approach?

Fermat's Idea

Suppose we can find x, y integers with $x^2 \equiv y^2 \pmod{N}$ and $x \not\equiv \pm y \pmod{N}$. Then $1 < \gcd(x - y, N) < N$ and this can be computed quickly, giving rise to a nontrivial factor of N.

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To find x and y, the most successful technique uses smooth numbers (divisible by "small" primes only). It is due to Morrison & Brillhart.
This idea is at the heart of the most successful factoring methods (QS and NFS), except ECM.

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- QS: $\exp\left(c_1(\log N)^{1/2}(\log\log N)^{1/2}\right)$
- ECM: $\exp\left(c_2(\log p)^{1/2}(\log\log p)^{1/2}\right)$, (where p is smallest prime dividing N)
- NFS: $\exp\left(c_3(\log N)^{1/3}(\log\log N)^{2/3}\right)$

Presentation of Current Work

We present an approach which is likely to yield

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- Translates an arithmetic problem into an analytic one.

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- It does not use the Morrison-Brillhart paradigm.
- It is in my view more natural, as it relates quantities known for their intrinsic arithmetical significance.
- Translates an arithmetic problem into an analytic one.
- All running times are proven, no assumptions!
- Much room for future improvements.

Approaching Multiplicative Functions

Let $\phi(n)$ be the Euler phi function. Suppose that N factors as $N = pq$, so that $\phi(N) = N - p - \frac{N}{p} + 1 = f(p)$.

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How do we find a good approximation to $\phi(N)$?

First Attempt with Riemann

Riemann zeta function is

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \quad \Re s > 1$$

It can be continued to a meromorphic function in $\mathbb{C}$ with simple pole with residue 1 at $s = 1$. Also

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n \geq 1} \frac{\phi(n)}{n^s} \quad \Re s > 2$$

Isolating $\phi(N)$

Classical: Compute $\Phi(x) = \sum_{n<x} \phi(n)$ by

$$\Phi(x) = \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} \frac{\zeta(s-1)}{\zeta(s)} \frac{x^s}{s}\, ds$$

and move line of integration "to the left".

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and move line of integration "to the left". Problem: we hit the Riemann zeros, spooky beings! Can we avoid them?

Second Attempt with Riemann

We now consider $\sigma(N) = N + 1 + p + q$. As before, a close approximation to $\sigma(N)$ will reveal p. Here

$$\zeta(s)\zeta(s-1) = \sum_{n \geq 1} \frac{\sigma(n)}{n^s} \quad \Re s > 2$$

and hence if $S(x) = \sum_{n<x} \sigma(n)$ we get

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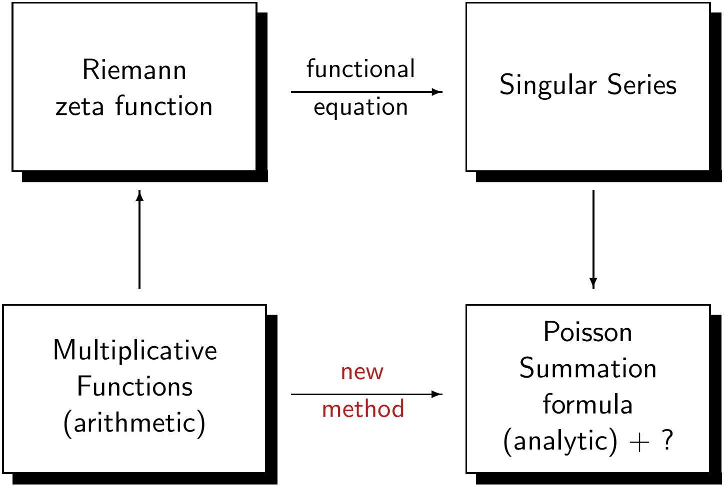
Problem: $|\zeta(s)| \approx |s|^{(1-\Re s)/2}$ as $|\Im s| \to \infty$ so cannot move the line of integration far enough to the left (to $\Re s \leq 0$)

The Mellin Transform Approach



The Multiplicative Function

We let $r \geq 2$ and $\beta_m \leq 0$ "fixed". Define

$$\sigma_r(n) = \sum_{d_1 d_2 \cdots d_{r-1} | n} d_1^{\beta_1} d_2^{\beta_2} \cdots d_{r-1}^{\beta_{r-1}}$$

Then

$$\zeta(s)\zeta(s-\beta_1)\cdots\zeta(s-\beta_{r-1}) = \sum_{n=1}^{\infty} \frac{\sigma_r(n)}{n^s}$$

The Test Function

We let $\nu \in \mathbb{R}$ with $\nu > 1$. Define

$$f_\nu(t) = \begin{cases} (1-t)^{\nu-1} & 0 \le t \le 1 \\ 0 & t \ge 1 \end{cases}$$

The Mellin transform of f_ν is

$$\frac{\Gamma(\nu)\Gamma(s)}{\Gamma(\nu+s)} = \int_0^\infty f_\nu(t) t^{s-1}\, dt$$

Inverse Mellin Transform

We have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\zeta(s)\zeta(s-\beta_1)\cdots\zeta(s-\beta_{r-1})\frac{\Gamma(\nu)\Gamma(s)}{\Gamma(\nu+s)}\,x^s\,ds = \sum_{n\le x}\sigma_r(n)f_\nu\left(\frac{n}{x}\right)$$

Call the right-hand side

$$F(\nu)=\sum_{n\le x}\sigma_r(n)\left(1-\frac{n}{x}\right)^{\nu-1}$$

Isolating p dividing N

We estimate

$$F^{(k)}(\nu) = \sum_{n \le x} \sigma_r(n) \left(1 - \frac{n}{x}\right)^{\nu - 1} \log^k \left(1 - \frac{n}{x}\right)$$

If $x = N + \frac{1}{N^2}$ we get

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If $x = N + \frac{1}{N^2}$ we get

$$F^{(k)}(\nu) = (-3 \log N)^k \sigma_r(N) N^{-3(\nu-1)} + O(N(2 \log N)^{k+r})$$

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Choosing $k > c_1(\nu + r) \log N$ and supposing we can compute $F^{(k)}(\nu)$ with good precision we get a value for $\sigma_r(N)$ up to an error $O(N^{-c_2})$, where $c_2 \to \infty$ as $c_1 \to \infty$. If $N = pq$, then as before this is sufficient to obtain p.

The Functional Equation

The Riemann zeta function is an meromorphic function with a single pole at 1 with residue 1 satisfying the functional equation (given here in asymmetric form)

$$\zeta(s) = \frac{(2\pi)^s}{\pi}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)$$

New Identities

Moving the line of integration to the left and using the functional equation shows

$$F(\nu) \approx \rho + \frac{(2\pi i)^{r-\beta_1-\cdots-\beta_{r-1}}\,\Gamma(\nu)}{(2\pi i)^{r+1}}\, x(\cos \pi\nu - \sin \pi\nu)$$
$$\times \int_{(1+1/r)} \{(2\pi i)^r x\}^{-s}\,\Gamma(s-\nu)\Gamma(s+\beta_1)\cdots\Gamma(s+\beta_{r-1})$$
$$\zeta(s)\zeta(s+\beta_1)\cdots\zeta(s+\beta_{r-1})\,ds$$

where ρ is some easily expressible residue. In view of the previous expression, it is appropriate to choose r so that $x^{1/r} \approx e$ so that the integral does not depend on x (hence N).

The Singular Series

Using the multiplication theorem

$$\Gamma(s)\Gamma\left(s+\frac{1}{r}\right)\Gamma\left(s+\frac{2}{r}\right)\cdots\Gamma\left(s+\frac{r-1}{r}\right) = (2\pi)^{(r-1)/2} r^{1/2-rs}\Gamma(rs)$$

we arrive at evaluating terms for $F^{(k)}(\nu)$ which consist of derivatives of $\Gamma(\nu)(\cos\pi\nu - \sin\pi\nu)$ times the following series

$$\sum_{d_1,\ldots,d_r\geq 1} \frac{d_1^{-\beta_1} d_2^{-\beta_2}\cdots d_{r-1}^{-\beta_{r-1}} \log^j d_r}{(d_1\cdots d_r)^2}\cdot e^{2\pi i r (d_1\cdots d_r)^{1/r}}$$

with $j \leq k = O(\log^2 x)$. Here $r = [\log x]$.

Computing the Singular Series

How do we evaluate the singular series? Find an algorithm which computes it within e^{-t} in $O(t^c)$ binary operations for some fixed $c > 0$. Can allow dependence on r to be bad, for instance e^r. In general, need to evaluate

$$\sum_{d_1,\ldots,d_r \geq 1} \frac{\log^j d_r}{d_1^{c_1} d_2^{c_2} \cdots d_r^{c_r}} \cdot e^{iAd_1^{a_1} \cdots d_r^{a_r}}$$

with $A = O\big(\max\big(1/a_1, \ldots, 1/a_r\big)\big)$ and $a_1 + \cdots + a_r = 1$.

Poisson Summation Approach

Compute

$$\sum_{d_1,\ldots,d_r \geq 1} e^{-\frac{1}{d_1^R}} \cdots e^{-\frac{1}{d_r^R}} \frac{\log^j d_r}{d_1^{c_1} d_2^{c_2} \cdots d_r^{c_r}} \cdot e^{iAd_1^{a_1} \cdots d_r^{a_r}}$$

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Poisson Summation Approach

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$$
\begin{aligned}
&\sum_{d_1,\ldots,d_r\geq 1} e^{-\frac{1}{d_1^R}} \cdots e^{-\frac{1}{d_r^R}} \frac{\log^j d_r}{d_1^{c_1} d_2^{c_2} \cdots d_r^{c_r}} \cdot e^{iAd_1^{a_1}\cdots d_r^{a_r}} \\
&= \sum_{d_1,\ldots,d_r\geq 1} f(d_1,\ldots,d_r) \\
&= \sum_{x_1,\ldots,x_r\in\mathbb{Z}} \hat{f}(x_1,\ldots,x_r)
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Estimate the rate of decrease of $\hat{f}$.

Poisson Summation Approach

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Estimate the rate of decrease of $\hat{f}$. Evaluate single coefficients.

Rate of Decrease of $\hat{f}$

$$\hat{f}(x_1,\ldots,x_r) = \int_{\mathbb{R}^r} e^{-\frac{1}{u_1^R}} \cdots e^{-\frac{1}{u_r^R}} \frac{\log^j u_r}{u_1^{c_1} u_2^{c_2} \cdots u_r^{c_r}} e^{iAu_1^{a_1}\cdots u_r^{a_r}} e^{-2\pi i(x_1u_1+\cdots+x_ru_r)}\, du_1\cdots du_r$$

We can modify contours ($u_m \to e^{\pm i\theta} u_m$ for $\theta > 0$) to get

$$\hat{f}(x_1,\ldots,x_r) = e^{\pm i\theta\star} \int_{\mathbb{R}^r} e^{-\frac{e^{\pm iR\theta}}{u_1^R}} \cdots e^{-\frac{e^{\pm iR\theta}}{u_r^R}} \frac{\log^j e^{\pm i\theta} u_r}{u_1^{c_1} u_2^{c_2} \cdots u_r^{c_r}} e^{ie^{\pm i\theta}Au_1^{a_1}\cdots u_r^{a_r}} e^{2\pi i e^{i\theta}(|x_1|u_1+\cdots+|x_r|u_r)}\, du_1\cdots du_r$$

Rate of Decrease of $\hat{f}$ (cont'd)

$$\hat{f}(x_1,\ldots,x_r) = e^{\pm i\theta\star} \int_{\mathbb{R}^r} e^{-\frac{e^{\pm iR\theta}}{u_1^R}} \cdots e^{-\frac{e^{\pm iR\theta}}{u_r^R}} \frac{\log^j e^{\pm i\theta} u_r}{u_1^{c_1} u_2^{c_2} \cdots u_r^{c_r}} e^{ie^{\pm i\theta} A u_1^{a_1} \cdots u_r^{a_r}} e^{2\pi i e^{i\theta}(|x_1|u_1+\cdots+|x_r|u_r)} \, du_1 \cdots du_r$$

therefore

$$\left|\hat{f}(x_1,\ldots,x_r)\right| \ll \int_{\mathbb{R}^r} e^{-\frac{\cos(R\theta)}{u_1^R}} \cdots e^{-\frac{\cos(R\theta)}{u_r^R}} \frac{\log^j u_r}{u_1^{c_1} u_2^{c_2} \cdots u_r^{c_r}} e^{\pm A(\sin\theta) u_1^{a_1} \cdots u_r^{a_r}} e^{-2\pi(\sin\theta)(|x_1|u_1+\cdots+|x_r|u_r)} \, du_1 \cdots du_r$$

Rate of Decrease of $\hat{f}$ (end)

A New Factoring Algorithm

Francesco Sica

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Introduction

Heuristics

Factoring with the Riemann zeta function

Conclusion

The cool case:

$$\begin{aligned} A(\sin\theta)u_1^{a_1}\cdots u_r^{a_r} &\le A(\sin\theta)(a_1u_1+\cdots+a_ru_r) \\ &\le (2\pi-\delta)(\sin\theta)(u_1+\cdots+u_r) \end{aligned}$$

i.e. if $A \le (2\pi-\delta)\min(1/a_1,\ldots,1/a_r)$ for some $\delta > 0$.

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i.e. if $A \le (2\pi-\delta)\min(1/a_1,\ldots,1/a_r)$ for some $\delta > 0$.
Problem: In our case

$$A = (2\pi + O(\log r/r))\min(1/a_1,\ldots,1/a_r)$$

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- Hope one can find a method to treat the seemingly innocuous but revealing

$$\sum_{n,m\geq 1} \frac{\cos(An^a m^{1-a})}{n^2 m^2}$$

within e^{-t^3} when $a = 1/t$ and $A = 3\pi$. In this case the best algorithm I found runs in $O(C^t)$ whereas if $A = \pi$ this is essentially $O(t^2)$.

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- Ultimately, this would show that factoring could be done in $O(e^r)$ for $r = \log^c N$ (subexponential).
- Need to perform extensive numerical calculations.

Conclusion

- Completely new (?) approach to factoring.
- Advantage is that it transforms the arithmetic problem of factoring N into a purely analytic one (evaluation of the singular series, "independent" of N).
- Should lead to a deterministic subexponential factoring algorithm with proven running time faster than what is currently known.
- Preprint available on ArXiv.

THANK YOU! ☺